COMPUTATIONAL MATHEMATICS AND ANALYSIS

COMBINATORICS
First Steps

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Contents

Preface

This book consists of three blocks. First, we introduce the basic principles and methods, which combinatorial calculations are based upon. The rule of product, the identity principle, recurrence relations, and inclusion-exclusion principle are the most important of the above. The method of generating functions is bypassed almost completely. This technique is too complex for an introduction to the subject.

A significant part of the book is devoted to classical combinatorial structures, such as ordering (permutations), tuples, subsets (combinations). A great deal of attention is paid to the properties of binomial coefficients, and in particular, to model proofs of combinatorial identities. We believe that in addition to their cognitive importance, such proofs are of remarkable aesthetic value. The third part of the book consists of the problems concerning some exact combinatorial configurations: paths in a square; polygonal chains constructed with chords of a circle, trees (undirected graphs with no cycles), etc. All chapters contain a considerable amount of exercises of various complexity: from easy training tasks to complex problems which require decent persistence and skill from the one who dares to solve them.

If one aims to passively familiarize oneself with the subject, methods and the most necessary facts of combinatorics, then it may suffice to limit one's study to the main text omitting the exercise part of the book. However, for those who want to immerse themselves in combinatorial problems and to gain skills of active research in that field, the exercise section is rather important. The authors hope that the book will be helpful for several categories of readers. University teachers and professors of mathematics may find somewhat unusual coverage of certain matters and exercises, which can be readily applied, in their professional work. We believe that certain series of problems may serve as a base for serious creative works and essays. This especially refers to students of pedagogical universities and colleges who need to prepare themselves for the teaching of basics of combinatorics based on school math courses, mainly building on arithmetic and geometry. Most of the exercises of the book are of this very origin. This book may prove useful for high school students and freshmen university students who are eager to make their first yet bold steps in mastering a new and exciting branch of mathematics. For these potential readers, there is a warning we need to give: the phrase "first' steps" mentioned above does not mean you will have a walk in the park. Be focused and persistent, prepare yourself for a long work which you will need to remain interested in. Do not despair if you initially fail to find the keys to some problems. This is normal. Move on coming back to such tough nuts, especially if they have you intrigued. Your award will be new knowledge, new skills, a pleasant feeling of having worked well done, and acquaintance with the basics of wonderful mathematical

science. Even in a pragmatic modern world, there are people (of various professions) who are in love with mathematics and prefer contemplating an interesting problem to pointless wasting of their lives in questionable fun. Such people are most interested in "short mathematical forms": efficient proofs, unbelievably simple solutions to what seems to be tough problems, enhancement of the existing solutions, the discovery of a relationship between very different by their form and sense facts. All of the above can be found in this book. The majority of such "mini-masterpieces" are now part of mathematical folklore. However, they were created by prominent mathematicians of different times: from Ancient Greece to nowadays. Finally, we believe this book will assist those brave men who dare to learn the basics of combinatorics to move on to its higher stages.

There is no special preliminary training required to understand any chapter. A regular school math course will suffice; it is enough to know the major facts from arithmetic, elementary algebra, and geometry. In addition, a vast amount of these facts is provided in the book for the comfort of the reader.

Introduction

The main goal of our book is to provide easy access to the basic principles and methods, which combinatorial calculations are based upon. The rule of product, the identity principle, recurrence relations, and inclusion-exclusion principle are the most important of the above. A significant part of the book is devoted to classical combinatorial structures, such as ordering (permutations), tuples, subsets (combinations). A great deal of attention is paid to the properties of binomial coefficients, and in particular, to model proofs of combinatorial identities. Problems concerning some exact combinatorial configurations: paths in a square; polygonal chains constructed with chords of a circle, trees (undirected graphs with no cycles), etc., are included too. All chapters contain a considerable amount of exercises of various complexity: from easy training tasks to complex problems which require decent persistence and skill from the one who dares to solve them. If one aims to passively familiarize oneself with the subject, methods, and the most necessary facts of combinatorics, then it may suffice to limit one's study to the main text omitting the exercise part of the book. However, for those who want to immerse themselves in combinatorial problems and to gain skills of active research in that field, the exercise section is rather important. The authors hope that the book will be helpful for several categories of readers. University teachers and professors of mathematics may find somewhat unusual coverage of certain matters and exercises which can be readily applied in their professional work. We believe that certain series of problems may serve as a base for serious creative works and essays. This especially refers to students of pedagogical universities and colleges who need to prepare themselves for the teaching of basics of combinatorics based on school math courses, mainly building on arithmetic and geometry. Most of the exercises of the book are of this very origin.

Keywords: combinatorics, rule of product, identity principle, recurrence relations, inclusion-exclusion principle, ordering, permutations, tuples, subsets, combinations, trees, graphs

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Chapter 1

Elementary Enumerations of Combinations

1. What is Combinatorics?

Arithmetic studies the properties of natural numbers and the principles of manipulating them, known as the arithmetic operations (addition, subtraction, multiplication, and division). Plane geometry (planimetric) provides an interpretation of important patterns concerning such shapes as triangles, circles, trapezia, parallelograms, etc. In addition, what does combinatorics deal with? Probably the best way to form the correct vision of the subject of combinatorics is through the consideration of specific examples from its domain.

Example 1.1. *Is there a way to place the numbers* $1,2,3,4,5,6,7,8,9$ *in a 3×3 square grid so that the sums of numbers in all rows, columns and diagonals are equal to the same value?*

Clearly, this is not a complex problem. After several efforts, one almost inevitably reaches the desired placing. For example, the following:

$$\begin{matrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{matrix}$$

Hence, the answer to the question is positive. Moreover, it yields another one, much less trivial question: how many such 3×3 square grids exist?

Example 1.2. *Let us have a drawing with small circles denoting cities and lines denoting routes between them. Departing from city A, is it possible to return to city A by traveling each route exactly once (cities may be revisited more than once)? The answer is positive for the provided scheme of routes. Moreover, this is true for any city in the drawing. Explain the reasoning behind that fact. Which special feature (or features) should a scheme obtain in order for the answer to remain positive? Come up with the easiest possible scheme, which does not allow a journey with stated conditions.*

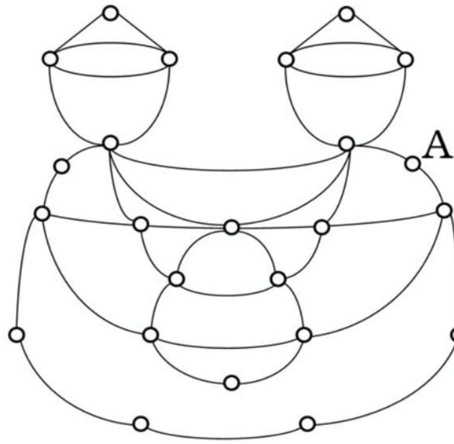

Figure 1.1. Travelling routes.

Example 1.3. *A vacuum tube has n plugs, placed as a circle and enumerated from 1 to n. It connects to a power unit with n sockets placed as a circle. Is there an enumeration of the sockets with the numbers from 1 to n which ensures that when the tube is plugged into the power unit, there is at least one plug inserted into a socket with the same number?*

This is a brilliant exercise from the Ukrainian Mathematical Olympiad. Prove that the answer depends on n being even or odd.

Example 1.4. *A domino tile covers two cells of a chessboard. Is it possible to construct a one-layer cover of an* 8×8 *chessboard with domino tiles so that all the cells are covered, except two cells laying in the diagonally opposite corners?*

Example 1.5. *How many regions can n lines split the plane into? How many regions can* ***not*** *n lines split the plane into?*

Is there anything in common with these five problems? At first glance, there is not. However, a more thorough analysis reveals that these problems are ideologically connected. All of them raise the question of existence or otherwise of a certain configuration: a grid of numbers with certain characteristics, a closed route on a scheme of routes, domino tiling, division of the space with n lines into a certain number of regions, etc.

Problems concerning the existence of configurations represent one part of the area of interest of combinatorics. The other part consists of the problems of listing the configurations that have already been proved or have been obvious from the start. Further, we will be predominantly interested in the problems of listing. We begin with some trivial yet instructive and characteristic examples of problems of this type.

Example 1.6. *How many different integer triangles having sides a and b exist? (an integer triangle is a triangle all of whose sides have lengths that are integers)*

To construct a triangle, we need one more side, say, x. The amount of triangles satisfying the stated conditions depends on the domain of values of x. From geometry, we

know that a triangle with sides x, a and b exists if and only if x is greater than the difference of other sides' lengths and smaller than their sum. Arithmetically this requirement is expressed as follows:

$$|a-b| < x < a+b.$$

Moreover, in our case x is supposed to be integer. Hence, the problem reduces to purely arithmetical one: how many integers are there between two given numbers m and n ($m = |a-b|$, $n = a+b$)? Obviously, the answer is provided by subtracting the lesser of them m from $n-1$.

Conclusion: There are $a+b-|a-b|-1$ different integer triangles having sides a and b.

The answer gets simpler form if we know which of the numbers a and b is lesser. Let $a \geq b$. Then $|a-b| = a-b$ and the wanted number of triangles is given by the formula $2b-1$. It is a curious fact that the number of triangles depends only on the length of the shorter side. For example, if $b=1$, then there is only one triangle (having sides $1, a, a$) for any a.

Exercise. Write down all possible values of the length of third side for the following cases: 1) $a=7$, $b=3$; 2) $a=1$, $b=4$.

Example 1.7. *How many three-digit natural numbers are divisible by* 3*?*

The following counting technique seems completely acceptable. First, we find how many numbers from 1 up to the largest three-digit number 999 are divisible by 3. Every third number is divisible by 3, so there are $999:3 = 333$ eligible numbers in the interval from 1 to 999. By analogy, there are $99:3 = 33$ numbers, which are divisible by 3, in the interval from 1 to 99. Hence, there are $333-33 = 300$ numbers among all three-digit numbers.

Can you suggest another counting approach?

Example 1.8. *There are n lines on the plane, any two of which intersect and any three do not have a common point. 1) How many points of intersection are there? 2) How many regions do these lines split the plane into?*

There are many various approaches to the counting of points of intersection. Below we outline three of them.

Approach I. Imagine that we draw lines one by one and with the mind's eye observe how the points of intersection appear. The report on our observations may look as follows:

The first line is drawn. There are no points of intersection.

The second line is drawn. As it isstated in the problem, this line crosses the first one. The first point of intersection has appeared.

The third line is drawn. It should cross two previous lines. Two more points of intersection have appeared.

By analogy, the fourth line adds three new points of intersection, because it necessarily crosses all three previous lines in new points.

The fifth line adds four points of intersection, the sixth – five, and so forth. Finally, the last one n-th line increases the number of points of intersection by $n-1$. Thus, in the end, we have

$$1+2+3+4+5+\cdots+(n-1)$$

points of intersection.

The summands here are consecutive terms of arithmetic progression, so the result is available in nice and compact form: $\frac{n(n-1)}{2}$.

Approach II. Analyze the complete combination of lines (all the lines are drawn in accordance with the conditions of the problem). The question is: how many points of intersection are there on one line? The answer is $n-1$, and additionally, it does not depend on the choice of the line. Why? Because each line crosses the other $n-1$ lines. Hence, there are n lines and each of them has $n-1$ points of intersection. However, the product $(n-1)\cdot n$ is not the wanted amount. The reason is that any point of intersection belongs to two lines, and therefore has been accounted for twice. This observation yields that the correct number of points of intersection is $\frac{n(n-1)}{2}$.

Approach III. Denote the number in question by x and attempt to solve the problem using the "universal Descartes method", which means to construct the equation for the unknown variable x. Let the lines be marked $l_1, l_2, \ldots, l_n$, and the points of interaction are $A_1, A_2, \ldots, A_x$. Consider all such pairs $< A_i, l_k >$, for which A_i belongs to l_k. How many such pairs are there? We have two ways to calculate this amount. For each point A_i there are two such pairs as A_i is created by the intersection of two lines. Hence, overall there are $2x$ pairs. On the other hand, for each line l_k there are $n-1$ such pairs because each line has $n-1$ points of intersection. Therefore, we have $n\cdot(n-1)$ pairs. Thus,

$$2x = n(n-1),$$

which yields

$$x = \frac{n(n-1)}{2}.$$

Note that the third approach is somewhat similar to the second one. However, it is still different in how strictly the strategy on construction of equations is adhered to. Now, we turn to the second part of the problem: how many regions do the lines split the plane into?

We outline two approaches to this problem.

Approach I. Again, imagine the lines drawn one after the other, and carefully observe how each successive line increases the number of areas. When there are no lines, we have one region the whole plane. The first line splits it into two parts. The second line adds two more regions – by the number of half-lines (rays) created from it by the point of intersection with the previous line since each of these half-lines splits the corresponding half-plane into two parts. The third line is divided by two intersection points into three intervals, so the number of regions increases by 3. The algorithm of "proliferation" of regions is now clear: the k-th line adds k regions. Hence, after the n-th line is drawn the plane is split into

$$1+(1+2+3+4+\cdots+n) = 1+\frac{n(n+1)}{2}$$

regions.

Approach II. It is different from the first one in its final phase. While the observations and facts remain the same, the interpretation is different, which results in a different path to the answer. Answering the question about the number of regions added by the k-th line, we focus our attention on the fact that this amount is greater by 1 than the number of points of

intersection laying on this line, while in the previous approach we focused on the fact that there are k such regions. Hence, we conclude: on each drawing stage, the amount of added regions is given by the sum of two numbers – the number of points of intersection created and the number of drawn lines. Therefore, after the n-th line is drawn we find the plane spilled into

$$1+\frac{n(n-1)}{2}+n$$

parts. The first summand is the solid plane, which we have at the beginning of the process, the second is the number of points of intersection, and the last one is the number of lines.

Example 1.9. *How many two-digit numbers can be written using even digits only?*

Here are all the even digits: $0,2,4,6,8$. *A two-digit number can not begin with 0. Amount of which of the wanted numbers is greater: those beginning with 2, or those beginning with 4 (or 6, or 8)? This question is rhetorical yet very important. It is rhetorical because the answer "the amounts are the same" is obvious. And it is important because the answer to this question solves the problem. Indeed, five numbers begin with 2 as there are five possible choices for the second digit. Hence, there are 20 numbers of the required type.*

Example 1.10. *A cube has faces of different colors. It is stored in a niche of exactly the same size as the cube. How many ways are there to insert the cube into the niche?*

There are six ways to choose the bottom face, on which the cube is supposed to lie. For each of these choices, there are four ways to choose the front face. Hence, overall there are $6\cdot 4=24$ *choices.*

Problems

Problem 1.1. *How many two-digit integers are there? Three digit?*

Problem 1.2. *How many positive integers are there among first thousand, which: 1) are divisible by* 5*? 2) are divisible by* 7*? 3) are not divisible by* 11*? 4) are divisible by* 11*, and are not divisible by* 3*?*

Problem 1.3. *How many positive divisors does the number* 3^{17} *have? The number* 5^{17}*?*

Problem 1.4. *How many integer points are laying in the interval: a)* $[-13,29)$*? b)* $(9,87)$*? c)* $[a,b]$*? d)* $(a,b]$*? e)* $[a,b)$*? (a and b are integers,* $a<b$*).*

Problem 1.5. *Consider points of the coordinate system. Let a point be called an integer point if it has integer-valued coordinates. A point belongs to the first quadrant if all its coordinates are non-negative.*

1. *How many integer points are there in the first quadrant sums of coordinates of which: a) equals n? b) less than n? (n - fixed positive integer).*

2. *How many integer points are there such that each coordinate is less than 25 and the sum of coordinates: a) is greater than or equal to 25? b) greater than 25?*

Problem 1.6. *How many two-digit numbers can be written using odd digits only?*

Problem 1.7. *How many two-digit numbers can be written using one even and one odd digit?*

Problem 1.8. *How many four-digit numbers are there that are divisible: a) by* 7*? b) by* 13*?*

Problem 1.9. *How many three-digit numbers can be written using exactly one zero?*

Problem 1.10. *How many three-digit numbers can be written with at most two different digits?*

Problem 1.11. *How many integer isosceles triangles have the longest side of: a)* 10*? b)* $2k$*? c)* $2k+1$*?*

Problem 1.12. *How many integer isosceles trapezia have lateral sides of* n *and bases less than* n*?*

Problem 1.13. *How many integer isosceles trapezia have a larger base* n *and lateral sides less than* n*?*

Answer. $\frac{(3n-2)n}{4}$, *if* n *is even;* $\frac{(3n+1)(n-1)}{4}$, *if* n *is odd.*

Problem 1.14. *A cube has edges of length 1. What is the length of the shortest route from vertex A to the opposite (the furthest) vertex B? How many different shortest routes exists?*

Problem 1.15. *Let* p *be a fixed integer positive number. How many even integer numbers lay inside the interval* $\left(\frac{p}{2}, p\right)$*? In other words, how many even integer solutions does the inequality* $\frac{p}{2} < x < p$ *have?*

Answer. $k-1$, *if* $p = 4k$; k, *if* $p = 4k+1$ *or* $p = 4k+2$; $k+1$, *if* $p = 4k+3$.

Problem 1.16. *(Additional exercise.) Prove without performing calculations: 1) for a given even* p *there are the same amounts of even numbers in the intervals* $\left(0, \frac{p}{2}\right)$ *and* $\left(\frac{p}{2}, p\right)$*; 2) for a given odd* p *there is the same amount of even numbers in the interval* $\left(\frac{p}{2}, p\right)$ *as the amount of odd numbers in the interval* $\left(0, \frac{p}{2}\right)$ *(and vice versa).*

Hint. Consider the symmetry of the interval $[0, p]$ with respect to its middle point $\frac{p}{2}$.

Problem 1.17. *How many integer isosceles triangles have a perimeter of* p*?*

Hint. Prove that the amount of such triangles is equal to the amount of even numbers in the interval $\left(\frac{p}{2}, p\right)$.

Problem 1.18. *On the plane, there are* n *parallel lines of one direction and* m *parallel lines of another direction. 1) How many points of intersection of these lines are there? 2) How many regions do these lines split the plane into? 3) How many of them are bounded (and what shapes do they have) and how many are not? (A part of the plane is called bounded if it can be placed inside some circle)*

Answer. *1)* mn*; 2)* $(m+1)(n+1)$*; 3)* $(m-1)(n-1)$ *bounded regions (parallelograms) and* $2(m+n)$ *unbounded.*

Problem 1.19. *On the plane, there is a bunch (a collection of lines sharing a common point of intersection) of $m+1$ lines and $n+1$ parallel lines, one of which belongs to the bunch (i.e., is one of the mentioned $m+1$ lines).*

How many regions do these lines split the plane into? How many of them are unbounded?

Answer. *There are $(m+1)(n+2)$ regions; $2(m+n+1)$ regions are unbounded.*

Hint. Unbounded regions can be counted as follows. Imagine that we have a large enough circle to cover all the bounded regions. Only the bounded parts of unbounded regions are covered by the circle, so there is exactly the same amount of unbounded regions outside of the circle as it has been before it appeared. But there are $2(m+n+1)$ rays sticking out of the circle, which split the outer part of the plane into $2(m+n+1)$ parts.

Remark. The above reasoning is not correct if $m=0$ (that is, there is no bunch at all). Why? What would be the answers in this case?

Problem 1.20. *A rook is the only piece on the chessboard and it is located in the square $a1$. One has three moves to move it to the square $h8$. How many different ways of doing that exist?*

Answer. *36.*

Problem 1.21. *a) How many ways are there to place two rooks of different colors on an empty chessboard so that they do not attack each other (according to the chess rules)?*

b) What is the answer if both rooks are of the same color?

Answer. *a)* $64 \cdot 49$.

Hint. Prove that the amount of safe squares for the black rook does not depend on the place of the white rook. b) Half as much.

Problem 1.22. *A solution to an equation with two unknowns x and y is not a single value, but a pair of values: a value of x and a value of y. Additionally, these values should be named in a special order: first, the value of x, and then the value of y. In order to stress that, we call them an ordered pair of values and not just two values. The first of them (the value of x) is called the first component of a solution (pair) and the second one is called the second component. For example, the solutions of the equation*

$$2x-3y=4$$

are pairs $(2;0)$, $(5;2)$, $\left(\frac{1}{2};-1\right)$ *etc., but not the pairs* $(0;2)$, $(2;5)$, $(7;4)$.

A solution to an equation is called natural if both its components are natural (integer and positive) numbers. Similarly, a solution is called integer if both its components are integer (positive, negative or zero) numbers.

1. *How many natural solutions are there to the equation:*

   a) $x+y=20$?

   b) $x+y=n$ *(n is known natural number)?*

   c) $2x+y=20$?

   d) $xy=48$?

2. *How many integer solutions are there to the equation:*

a) $|x|+|y|=20$?
b) $|x|+|y|=n$?
c) $|x|y=48$?
d) $|x|\cdot|y|=48$?
Answer. *1. a)* 19*; b)* $n-1$*; c)* 9*; d)* 8*. 2. a)* 40*; b)* $4n$*; c)* 16*; d)* 32.

Problem 1.23. *There are n points on the plane any three of which do not belong to the same line. How many lines can be drawn when connecting these points pairwise?*
Answer. $\frac{1}{2}n(n-1)$.

Hint. First, find how many of the wanted lines pass through any one of the points.

Problem 1.24. *How many isosceles integer triangles are possible with the longest side (or one of the longest sides) of n?*
Answer. There are $2n-\left[\frac{n}{2}\right]-1$ *such triangles, where* $\left[\frac{n}{2}\right]$ *denotes integer part of the number* $\frac{n}{2}$.

Hint. Count triangles with the longest side being leg or base separately.

Problem 1.25. *How many integer triangles are there with the lengths of their sides forming an arithmetic progression and the shortest side of a?*
Name all the triplets of sides of such triangles for the case $a=7$.
Answer. $a-1$.

Hint. The triangle inequality yields the limitation for the common difference of arithmetic progression. What is this limitation?

Problem 1.26. *How many integer triangles are there with the longest side of a and the middle side of b (*$a \geq b$*; a and b are natural numbers)?*
Answer. $2b-a$.
Name all the triplets of sides of such triangles for the case $a=9$, $b=7$.

Problem 1.27. *How many parts do three chords split a circle into? Produce drawings illustrating all possible cases.*
Answer. Into 4, 5, 6 *or* 7 *parts.*

Problem 1.28. *How many parts can three lines split the plane into? Produce drawings illustrating all possible cases.*
Answer. Into 4, 5, 6 *or* 7 *parts.*

Problem 1.29. *Are the following amounts connected with each other: the amount of chords drawn in a circle, the amount of points of intersection of these chords inside the circle, the number of parts that these chords split the circle into?*
Answer. The last number is less than or equal to the sum of the first two. Under which conditions there is an equality?

Hint. If an inner point of a circle is a point of intersection of k chords, then the number $k-1$ is called a rank of this point. Prove that the amount of parts into which chords split a circle is equal to the sum of ranks of points of intersection of these chords plus the amount of these chords plus one.

Problem 1.30. *How many parts do the following objects split the plane into 1) n concentric circles; 2) n circles, any two of which do not have common points; 3) n circles, which have two common points?*

Answer. 1) $n+1$*; 2)* $n+1$*; 3)* $2n$.

Problem 1.31. *Circles are placed on the plane in such a way that: 1) all of them have the common pointA; 2) any two of them have a common point in addition to the point A; 3) any three of them do not have common points except for A. There are n circles. How many parts do these circles split the plane into?*

Answer. $1+\frac{n(n+2)}{2}$.

Sketch of Solution. Denote the wanted amount by t_n. We have $t_1 = 2$, $t_2 = 4$, which is easy to verify by drawing one and two circles. Assume there are k circles drawn. They split the plane into t_k parts (according to our notation). Observe how the amount of parts changes when the $(k+1)$-th circle is drawn. The new circle intersects with the previously drawn circles at $k+1$ points: at the point A and at k other points – one point belonging to each of those circles. These points split the new circle into $k+1$ arcs, and each of these arcs splits the previously solid part of the plane into two parts.

We reach the following conclusion:

If the circles are drawn one by one, then the s-th circle increases the amount of parts of the plane by s:

$$t_s = t_{s-1} + s. \tag{1.1}$$

Taking into account that in the beginning there is one solid plane and the first circle adds one new part, we have the following equalities:

$$\begin{aligned}
&t_0 = 1,\\
&t_1 = t_0 + 1,\\
&t_2 = t_1 + 2,\\
&t_3 = t_2 = 3,\\
&\dots\dots\dots\dots\dots\\
&t_n = t_{n-1} + n.
\end{aligned}$$

Adding these equalities term-wise (adding separately the left- and right-hand parts) we arrive at the wanted result:

$$t_n = 1 + (1+2+3+\ldots+n) = 1 + \frac{n(n+1)}{2}.$$

Evidently, the key instrument in the solution was formula (1.1). Formulas of such type are called recurrent. We will discuss this type of formulas in detail below.

Problem 1.32. *Which is the maximum attainable number of parts that n circles can split the plane into? How the circles should be drawn in order to reach this?*

Answer. $2+n(n-1)$.

Hint. We can follow the algorithm presented in the solution of the previous problem. This time we need to assure that each subsequent circle adds as many new parts of the plane as possible. This is the case when it intersects the previously drawn circle in the maximum possible number of points. If there are $k-1$ circles, then the k-th circle can add at most $2(k-1)$ points: two points per each previously drawn circle. This is the rule to follow while drawing circles one by one. Denoting τ_k the necessary amount of parts of the plane, which are produced by k properly drawn circles, we have the following formula

$$\tau_k = \tau_{k-1} + 2(k-1)$$

that provides the path to solution. This recurrent formula is valid for $k = 2$ onwards. Taking into account that $\tau_1 = 2$, we get the following equalities:

$$\begin{aligned}
&\tau_1 = 2, \\
&\tau_2 = \tau_1 + 2, \\
&\tau_3 = \tau_2 + 4, \\
&\tau_4 = \tau_3 + 6, \\
&\dots\dots\dots\dots\dots\dots \\
&\tau_n = \tau_{n-1} + 2(n-1).
\end{aligned}$$

Adding them term-wise we get

$$\tau_n = 2 + n(n-1).$$

Problem 1.33. *How many three-digit numbers are there, with the first and the third digits being equal, and the second digit is greater than the others?*

Answer. 36.

Problem 1.34. *How many three-digit numbers are there, with the first and the third digits being equal, and the second digit is less than the others?*

Answer. 45.

Problem 1.35. *How many three-digit numbers are there, with the second digit being the sum of the first and the third digits?*

Answer. 45.

Problem 1.36. *Bus tickets are enumerated by four digits from* 0000 *to* 9999 *inclusive.*

1) A ticket is deemed to be extremely lucky if its number is the same when reading it left to right and right to left. How many extremely lucky tickets are there?

2) A ticket is deemed to be lucky if the sum of the first two digits in its number equals to the sum of the last two digits. How many lucky tickets are there?

Answer. 1) 100; *2)* 508.

Solution of Problem: 2). If $0 \le k \le 9$, then there exist $k+1$ (ordered) pairs of digits, sum of which equals to k:

$$\begin{aligned}
&0+k=k,\\
&1+(k-1)=k,\\
&2+(k+2)=k\\
&\dots\dots\dots\dots\dots\dots\\
&(k-1)+1=k,\\
&k+0=k.
\end{aligned}$$

For example, if $k=4$, then there are five such pairs, namely: $04, 13, 22, 31$ and 40. Each such pair in the first two places of the number combines with each such pair in the last two places. Thus, there are 5^2 tickets with the "lucky" sums $(4;4)$. And when the sums are $(k;k)$, $0 \le k \le 9$, then there are $(k+1)^2$ lucky tickets.

Now, let $abcd$ and $\alpha\beta\gamma\delta$ be two numbers of tickets such that $a+\alpha=9$, $b+\beta=9$, $c+\gamma=9$ and $d+\delta=9$. If $a+b=c+d=k$, then $\alpha+\beta=\gamma+\delta=18-k$, and otherwise. This yields that the amount of numbers with the lucky sums of $18-k$ is the same as the amount of numbers with the lucky sums of k. For instance, there is the same amount of numbers with the lucky sums of 14 as there is with the lucky sums of 4.

The above facts provide that there are

$$2\cdot(1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2)+10^2$$

lucky tickets.

Problem 1.37. *How many natural solutions $(x;y;z)$ do the system of equations*

$$\begin{cases} x+y=m,\\ y+z=n \end{cases}$$

have (m and n are given natural numbers)?

Answer. *The lesser of numbers $m-1$ and $n-1$.*

Problem 1.38. *How many natural solutions $(x;y;z;t)$ do the system of equations*

$$\begin{cases} x+y=a,\\ z+t=b \end{cases}$$

have (a and b are given natural numbers)?

Answer. $(a-b)(b-1)$.

Problem 1.39. *How many natural solutions do the equation*

$$2x+y=n$$

have (n is given natural number)?

Answer. $\left[\frac{n-1}{2}\right]$, *which denotes the integer part of the number $\frac{n-1}{2}$.*

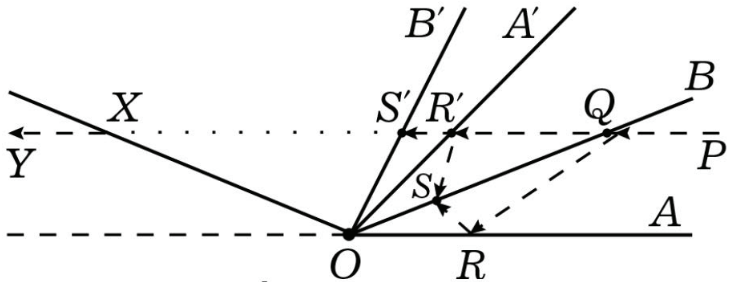

Figure 1.2. Reflections of a ray.

Problem 1.40. *Let the sides of the angle reflect rays of light (by the law of reflection, which states that the angle of incidence equals the angle of reflection). A ray, which is parallel to one side of the angle, strikes the other side. How many reflections from the sides of the angle will this ray have, if the angle is* 10° *?*

Answer. 17.

Solution. Let AOB be the 10° angle, formed by "linear mirrors" AO and BO. When a ray of light parallel to the side OA strikes the side OB it reflects in turn from both sides of the angle drawing a polygonal chain, the first segments of which are PQ, QR, RS, ... The statement of the problem is about the fate of this polygonal chain, or more exactly, the fate of the ray which forms it: will it get infinitely close to the point O making each time smaller and smaller jumps from one mirror to the other? Or after several reflections getting it closer to the point O it will start distancing from it? In this case, there is one more question: will the ray reflect infinitely from the mirrors, or after a certain number of reflections will it follow the straight line similar to the one, which has brought it into the angle? Finally, assuming the latter situation takes place, we need to determine how many reflections there are, that is, how many vertices its trajectory has.

Naturally, in order to answer the above questions, one needs to be acquainted with the law of reflection. Moreover, for someone who is not aware of this law, these questions may appear to be senseless. Recall the law of reflection: a ray reflects from the mirror at the same angle as it strikes it (see Fig. 1.3).

If there were a hole in the point of incidence, then the continuation of the ray's route would be symmetrical to the reflected ray. Taking into account this property, one can come up with quite smart answers to all the questions concerning the wandering of a ray of light between two mirrors. Consider Fig. 1.2. Let $\angle AOB = \angle BOA' = \angle A'OB' = \ldots.$ Assume that the sides of all these angles having the same vertex O can reflect the light. If the ray PQ from the statement of the problem is put through the hole into the angle BOA', then its route $QR'S\ldots$ inside this angle will be the symmetrical reflection with respect to OB of its route inside the angle AOB. In this case, the first segment QR' of the ray threaded through the hole Q is the (straight line) extension of the segment PQ. Similarly, if the ray PR' is threaded into the angle $A'OB'$ through the hole R', then its further trajectory in this angle is the symmetrical reflection with respect to OA' of its trajectory inside the angle BOA'. And the first segment of the chain $R'S'\ldots$, which forms in the angle $A'OB'$, corresponds to

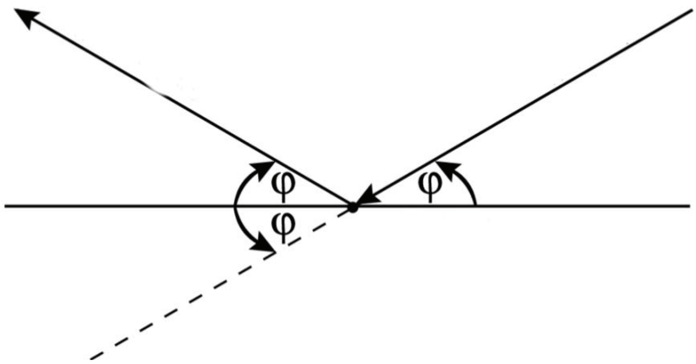

Figure 1.3. Law of reflection.

the second segment of the chain $QR'S\ldots$ placed in the angle BOA' and corresponds to the third segment of the primary chain $PQRS\ldots$ from the angle AOB. What will this exquisite procedure of straightening of the light chain $PQRS\ldots$ result into? Evidently, we will get the infinite ray xy placed in the eighteenth 10^0 angle, which in combination with all the preceding angles forms a straight angle. The points of intersection of the straight line PY with the half-lines OB, OA', OB', ... correspond to the vertices of the chain $PQRS\ldots$, and the intervals QR', $R'S'$, etc., correspond (precisely, preserving the distances) to its segments. In other words, the straight line PY with the points $Q, R', S', \ldots, x$ on it is the straightened variant of the polygonal chain $PQRS\ldots$. It is clear that there are 17 points $Q, R', S', \ldots, x$. Hence, this number of times the ray $PQRS\ldots$ reflects from the sides of the angle ABC. As an additional conclusion comes to the fact that at first the chain $PQRS\ldots$ goes "deeper" inside the angle AOB (in the direction of its vertex), but afterward walks outside, and finally, stops striking its mirror sides becoming a straight half-line.

Additional Exercise. Determine how many times the ray $PQRS\ldots$ reflects from the sides of 7° angle AOB.

Problem 1.41. *Each side of a triangle is split into n equal parts. Two lines are drawn through each splitting point parallel to the sides of the triangle, which do not contain this point. What are the shapes of regions into which these lines split the triangle? How many such regions are there?*

Answer. *Into n^2 identical triangles similar to the given one.*

Solution. Like many other exercises, this problem can be solved in several ways. This is true for the geometric part of the problem and the combinatorial one as well. We outline one of the possible solutions. Let ABC is the given triangle with a web of intervals inside it, which appeared as a result of the procedure described in the statement of the problem. Extend this triangle to the parallelogram $ABA'C$, replicating also the web from the original triangle in the triangle $BA'C$. Now, remove from the parallelogram $ABA'C$ all lines, which are parallel to the diagonal BC, including the diagonal itself. We are left with intervals parallel to the sides of the parallelogram. They are splitting it into n^2 identical parallelograms similar to the large one. If we return the removed intervals now, they will split each of

the small parallelograms into two identical triangles similar to the triangle ABC (or $BA'C$). There will be $2n^2$ small triangles, and half of them will fall into the triangle ABC.

Problem 1.42. *Each side of the parallelogram is split into n equal parts. Two lines are drawn through each splitting point parallel to the diagonals of the parallelogram. The diagonals are also drawn. What are the shapes of the regions into which these lines split the parallelogram? How many such regions are there? Count the figures of different shapes separately.*

Answer. Parts touching with one of their sides any side of the parallelogram are triangles. Overall, there are $4n$ of them, - n near each side. Other parts are parallelograms, and there are $2n(n-1)$ of them.

Solution. The number of small parallelograms can be determined as follows. Let O be the point of intersection of the diagonals of the given parallelogram $ABCD$. Construct a parallelogram from the triangles BOC and AOD joining their sides BC and AD. It appears that this parallelogram is fragmented into n^2 small parallelograms by the intervals of drawn lines. Those n among them, which are crossed by the side (that is now a diagonal) BC (or AD), are constructed of two triangles each. Others have been parallelograms since the beginning. Hence, there were $n^2 - n$ parallelograms. There is the same amount of parallelograms inside the triangles AOB and COD. There are $2n(n-1)$ altogether.

Problem 1.43. *The sides AB, BC and CD of the square $ABCD$ are split into n equal parts each. Splitting points of the sides BC and CD are connected by line segments with the vertex A, and splitting points on the sides AB and BC, – with the vertex D. Additionally, both diagonals of the square are drawn. 1) How many line segments are drawn including the diagonals? 2) How many parts do they split the square into? 3) At how many points do they intersect inside the square? 4) Is there a connection between three numbers, which are the answers to the previous questions? 5) Among the parts, into which the intervals split the square: a) how many triangles are there? b) how many quadrilaterals are there? 6) Will the answers to questions 1, 2, and 3 change if the word "equal" is omitted in the statement of the problem? 7) Will the answers to questions 1, 2, and 3 change if the square is replaced by: a) rectangle? b) random convex quadrilateral?*

Answer. 1) $4n-2$; 2) $\frac{1}{2}n(7n+1)$; 3) $\frac{7}{2}n(n-1)+1$; 4) Sum of the first and the third numbers is less than the second number by one. Find a combinatorial reasoning for that connection. 5) a) $5n-1$; b) $\frac{(n-1)(7n-2)}{2}$. 6), 7) Will not change.

Hint. Triangles are concentrated near the vertices A and D ($4n-1$ altogether) and near the side BC (n triangles). The rest of the parts are quadrilaterals and their number can be determined by subtraction.

2. Combinatorial Rule of Product

Behind this solid name, there is simple content, and the simplicity hides pitfalls which a beginner utterly needs to learn to bypass.

Example 1.11. *John eats in a café every day and every time follows the same rule: his meal consists of one entrée and one main course. There is a choice of five entrées and seven main courses today. How many options are there for John to configure his meal?*

The problem can be stated in a different way changing the emphasis in the question. Assume there are always the same five entrees and seven main courses on the menu. How many days can pass with John choosing a new combination for his meal?

There is no doubt the reader has already found an answer. However, taking into account that the situation in the problem may arise in different variations in the future, and the necessity to recognize it in more complex cases, we outline the details of the explanation of the answer.

Let us adhere to the second formulation of the question. Assume John decided to use the following algorithm. He is going to choose the same entrée adding variability to his meals by the choice of the main course. How many days John can choose meals without repetition? Obviously, the answer is seven. On the eighth day, he has to change the entrée. John will have another seven days of different meals with this choice of entrée. The same will happen for the other three choices of entrée. Hence, having five entrees and seven main courses he can choose $5 \cdot 7 = 35$ different meals.

Example 1.12. *How many two-digit numbers comprise odd digits only?*

The answer to the question can be illustrated by Fig. 1.4. The first row and the first column of the table consists of all five odd numbers each. Consider the square circled by the double line. Every cell can be specified by two numbers: first, the one placed to the left from it, and then, the one above it. Putting these numbers next to each other, we get a two-digit number, which can be taken as a code of the corresponding cell. Thus, every cell has a code attached to it, and every code denotes a specific cell. For example, the crosshatched cell has code 57. The cells are geometric analogs of their codes, which are two-digit numbers comprising of odd digits. Hence, the numbers of codes (two-digit numbers) and cells (their geometric analogs) are equal. The latter amounts to $5 \cdot 5 = 25$ (five rows with five cells in each). Therefore, there is the same amount of two-digit numbers, which is the answer to the problem.

	1	3	5	7	9
1					
3					
5				///	
7					
9					

Figure 1.4. Two-digit numbers.

A square (or a rectangle in general) divided into the required number of cells is a good model illustrating the rule of product. For a rectangular model, it reads as follows:

If a rectangle is divided into n horizontal (parallel to one pair of sides) and m vertical (parallel to the other pair of sides) stripes, then their crossing consists of $n \cdot m$ rectangular parts.

Instead of the rectangular model presented in the solution to Problem 2 (and similar problems), one can consider the "web" model, which does not differ from the former essentially. We draw five "horizontal" lines on the plane and enumerate them with digits $1,3,5,7$ and 9. Now, repeat the procedure drawing "vertical" lines this time. Five horizontal and five vertical lines cross, giving 25 points of intersection, which might serve as a "natural" analogs of those two-digit numbers both digits of which are odd. Because defining such a point is the same as defining those two lines (one horizontal and one vertical) intersection of which this point is.

Using the "web" model, we can present the rule of product in the following form:

If there are n horizontal (parallel to each other) lines and m vertical (also parallel to each other) lines, then their crossing produces nm points of intersection. Both these models, the rectangular one and the "web" model, efficiently illustrate and explain the answers $5 \cdot 7$ to the first problem and $5 \cdot 5$ in the second.

Now, we present another model, which unlike two previous models easily lends itself to generalizations.

Consider again the second variation of the formulation of the first problem. So, we assume that the café is reluctant to vary its menu and every day offers the same five entrees and seven main courses. And John wants to have a different meal every day for as long as possible. Moreover, John likes surprises and wants to make some for himself. To this end, he arranged some sort of a lottery. He wrote down all possible combinations of his meal on separate cards of the following form 1.1.

Table 1.1. Combinations of meal

Soup	Pasta
Cesar salad	Fish
...	...

Every time before the dinner, John randomly chooses one of the cards, orders the combination written on it, and throws the card away. How many cards are there? There are seven cards with the word "soup", seven cards with the word "Cesar salad", and three more times like that. Hence, there are $5 \cdot 7$ cards. One could count another way: there are five cards with the word "pasta", five cards with the word "fish" and five more times like that; so overall there are $7 \cdot 5$ cards.

The situation is similar to the second problem. To write down a two-digit the number is to place next to each other two digits. According to the statement of the problem, both of these digits should be odd. Hence, there are five choices for the first position (a digit denoting tens), and not depending on the actual digit chosen, there are five choices for the second digit (a digit denoting ones). The conclusion is that there are $5 \cdot 5$ ways to write down a two-digit number.

Both of these problems, together with all similar problems, can be illustrated by the following abstract scheme. Suppose we have a stripe divided into two cells or two cells placed next to each other (see 1.2). If we are allowed to place one of m available symbols into the first cell, and one of n symbols into the second cell (it does not matter whether the

Table 1.2. Stripe with two cells

sets of allowable symbols are the same), and the choice for the first cell does not affect the number of allowable choices for the second one, then there are $m \cdot n$ different ways of filling in both of these cells.

Example 1.13. *How many two-digit numbers are there with both digits being different?*

Obviously, one can get the answer by subtracting the amount of two-digit numbers comprising two same digits (nine) from the overall amount of two-digit numbers (90). However, using the rule of product is also an option. Think of this problem in the following way. In order to write down a number (one of those which are asked about in the statement), we need to decide upon the first digit (a digit denoting tens). There are nine choices for it (any one of ten available digits except zero). Now, there is a crucial question: is it correct that the amount of choices for the second digit (a digit denoting ones) does not depend on the first digit actually chosen? If the answer is positive, then the principle (rule) of product comes into play, and we immediately get the solution to the problem. However, if it is not the case, then we can not apply the rule of product, and we need to consider another way of solving the problem.

So we have chosen the first digit. How many possibilities are there for the second digit? The problem states that the second digit should differ from the first one. This is the condition that needs to be satisfied while choosing the second digit. Hence, there are nine choices for the second digit disregarding the choice made on the first step. The digit denoting ones could be any digit except the one, which has already been chosen to denote tens. The conclusion is that we can apply the rule of product. There are $9 \cdot 9 = 81$ wanted numbers.

Example 1.14. *How many two-digit numbers are there, which are divisible by 3 and can be written down using digits* $0, 1, 2, 3, 4, 5, 6$*?*

As the wording of the problem is very similar to the previous problems, it is reasonable to attempt solving it with the presented above method. There are six possible choices for the first digit (digit denoting tens) because it can be any of the mentioned in the statement digits except zero. Assume, we have chosen one of the digits $1, 2, 3, 4, 5$ or 6. How many ways are there for us to complete the number? Choosing the second digit, we should only care that the resulting number is divisible by 3. Recall the requirement for divisibility by 3: a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Therefore, we have the following:

1. if the first digit is 1 or 4, then the second one is 2 or 5;

2. if the first digit is 2 or 5, then the second one is 1 or 4;

3. finally, if the first digit is 3 or 6, then the second one is $0, 3$ or 6.

It appears that the amount of possible choices for the second digit depends on the value of the first digit. If the first digit is $1, 2, 4$ or 5, then there are two possibilities for the second digit; in the case, the first digit is 3 or 6, there are three choices for the second one: $0, 3$ or 6. This means that the rule of product is not applicable here.

In fact, this problem can be solved using the rule of product, if we apply the reverse algorithm. Begin with the second digit. There are 7 choices for it, as any digit from the statement of the problem, fit well. Now, let the second digit be chosen. When choosing the first digit, we need to make sure that the resulting number is divisible by 3 :

1. if the second digit is 1 or 4, then there are two options for the first one: 2 or 5;
2. if the second digit is 2 or 5, then there are still two options for the first one: 1 or 4;
3. if the second digit is $0, 3$ or 6, then there are two options again for the first one: 3 or 6.

As we can see the number of possibilities (which is two) for the first digit does not depend on the second one. This is what we look for when assessing the applicability of the rule of product. Hence, there are $7 \cdot 2 = 14$ numbers of interest.

This instructive (yet elementary) problem. It teaches that sometimes the applicability of the of the rule of product depends on the order of actions.

The rule of product is easily generalized from two positions to three or more. For example, let the question be about the number of triplets of symbols for which the following conditions are satisfied.

Table 1.3. Triplet of symbols.

a	b	c

1. the first symbol of the triplet (or its first component) a can be chosen in m different ways (in other words, the first position is for one of m predetermined symbols);
2. for any chosen value of the first component, the second component is chosen from n possibilities with no restrictions;
3. for any chosen values of the first and second components, we have k options for the third one.

Under such conditions, we can construct the pair in $m \cdot n$ ways. This results from the rule of

Table 1.4. Duplet of symbols.

a	b

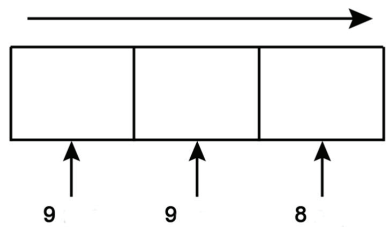

Figure 1.5. Three-digit numbers. A.

product for two positions.

After that, one can consider the triplet $(a; b; c)$ as a combination of the pair $(a; b)$ and the component c. There are $m \times n$ possibilities for the pair $(a; b)$. Under any choice of the pair $(a; b)$, we have k options for the component c. Hence, there are $mn \cdot k$ ways to construct the triplet $(a; b; c)$. Thus, under conditions 1) 2) and 3) there are mnk options for the triplet $(a; b; c)$. This is the rule of product for three positions (or for the objects having 3 components). There is no sense to provide the formal extension of the rule of product on the objects with $4, 5$ or more generally s components, as the procedure is similar to the algorithm of extension from two- to three-component objects. Below we consider several examples illustrating the application of the rule.

Example 1.15. *How many three-digit numbers are there with:*

1. *all digits being different;*
2. *any two adjacent digits being different?*

Begin with the first question. Let the number be written down digit by digit from left to right. We can begin with any digit except for zero. Hence, there are 9 options for the first digit (the digit denoting hundreds). Now, let the first digit be chosen. Does the number of options for the second digit depend on the actual choice of the first one? The answer is negative, as we can choose any digit except the one chosen for the first position. Hence, there are also 9 options for the second digit. Let the second digit is chosen. How many "degrees of freedom" do we have when choosing the last digit? Does the answer depend on the actual choice for the previous digits? These questions are crucial as the answers to them define if the rule of product is applicable in this situation. There is no doubt, the answer to the last question is negative: no matter which are the first two digits, there are still 8 options for the third one, as it may be any digit except for the one chosen previously. Let us make a conclusion. Moving left to right from one digit to another, we have found out that the number of options for a digit does not depend on the digits chosen previously. This means that the overall amount of numbers can be defined using the combinatorial rule of product. Therefore, the answer to the first question of the problem is $9 \cdot 9 \cdot 8$.

It is worth noting that the rule of product does not work in this example if we move from right to left. In order to make sure about this fact and understand reasoning behind it, we start from the last digit of a number. It is obvious, that we have all ten digits to choose from. Assume, the choice of the last digit is made, and we have actual value in the place of ones of a number. Does it affect the amount of options for a digit to be put in the place

Figure 1.6. Three-digit numbers. B.

of tens? The answer is no. Really, any digit can fit the next place provided it differs from the one already chosen. Hence, there are 9 options for the place of tens under any choice of the third digit. Assume now, we have chosen two last digits, and thus realized one of 90 possibilities. Let these digits be a (in the place of ones) and b (in the place of tens). How many options are there for the first digit? Recall that $b \neq a$ and the digit denoted by the question mark differs from a and b. So how many possibilities are there for the question mark? It appears that we can't answer this question. We can state any actual digit: 8, or 7, or some else digit. Why? Because the answer to this question strongly depends on a and b. If there is no zero among them, then one of 7 digits can be put in the place of hundreds: not a, not b and not zero (as the problem is about three-digit numbers. Alternatively, if $a = 0$ or $b = 0$, then there are eight options for the digit to be put instead of the question mark: not a and not b. Thus, we are in a situation when the rule of product fails to help us.

We have ensured once again that the "straight" and "reverse" approaches to the enumeration of multicomponent objects are not identical (or more precisely, not always identical). Particularly, the order of selection often affects the applicability of the rule of product.

The second question of the problem can be dealt with in a similar manner. There are 9 choices for the place of hundreds. Realizing any one of them, we have again 9 choices for the second digit (any of ten digits except for the one chosen before). Finally, for any choice of the first two digits, there are 9 options for the third one (any of ten digits except for the one chosen for tens). Hence, there are $9 \cdot 9 \cdot 9$ three-digit numbers with the second digit being different from the two others.

Example 1.16. *Consider five-digit numbers having simultaneously the following properties:*

1. *decimal form of a number does not contain zeroes;*
2. *following an even digit, there is always a digit, which denotes a prime number (recall that a number is called prime if it has exactly two natural divisors);*
3. *there is always an even digit following an odd.*

How many such five-digit numbers are there?

It is always tempting to check if the rule of product can be applied. Begin with the left-hand side. Any digit except zero can be in the first place. Let the first digit be chosen. How many options are there for the second one? If the first digit is odd, then it must be followed by a digit being a prime number. There are 4 such digits, namely: $2, 3, 5$ and 7. Hence, there are 4 options for the second digit in this case. Alternatively, if the first digit is odd, then the next digit ought to be even, and not zero. Therefore, for the second place, we again have 4 possibilities. This is an encouraging fact. This is the property, which enables

one to apply the rule of product. If only it expanded for the remaining digits of a number. And this is the case! Let a be a digit in a random place of a number except for ones. Which digits can stand right after a? If a is even, then it could be $2, 3, 5$ or 7, and no others. If a is odd, then it should be followed exclusively by $2, 4, 6$ or 8.

Now, imagine, that we write down one of the numbers in question. Begin with the left-hand side, and write down the first digit, then the second one, the third, fourth and fifth. Each time, before writing one of the digits we need to choose it. Herewith, our choice on each stage will be to a certain extent regulated by the statement of the problem. Which freedom of choice will we have on each of the five stages? How many options will be there to choose from? The previous considerations provide the answer to these questions. We will choose the first digit out of nine possibilities, and each following – out of four. This circumstance enables us to use the rule of product. Hence, there are $9 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ numbers of interest.

Example 1.17 (About divisors of a number)**.** *Hereinafter we are talking about natural (integer and positive) divisors of natural numbers.*

Different numbers have different amounts of divisors. In the attached table, there are several natural numbers in the first row and their divisors in the second (under each number, there are all its divisors):

Table 1.5. Divisors (a)

Number	1	2	4	5	6	11	12
Divisor	1	1,2	1,2,4	1,5	1,2,3,6	1,11	1,2,3,4,6,12

Table 1.6. Divisors (b)

Number	15	17	20
Divisor	1,3,5,15	1,17	1,2,4,5,10,20

Giving at least cursory attention to the table, one can realize that the amount of divisors of a number has little dependence on its size. The numbers, which differ just a little (for example, by one) can have drastically different amounts of divisors. This observation could have obtained additional evidence if we were to expand the table. For instance, the number 100 *has* 9 *divisors* $(1, 2, 4, 5, 10, 20, 25, 50$ *and* $100)$*, and the number* 101 *has only two (*1 *and* 101*). It is obvious that every number except the number 1 has at least two divisors. If the number a has only two divisors then these are* 1 *and a. Numbers having only two divisors are called prime numbers. If we were to extract the sequence of prime numbers from the sequence of natural numbers, then the former would start as follows*

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots$$

Non-prime numbers, which are greater than one, have at least two divisors. Here is the

table of some of the first natural numbers, providing the amount of divisors of each number below it.

Table 1.7. Amounts of divisors (a)

Number	1	2	3	4	5	6	7	8	9	10	11	12
N of divisors	1	2	2	3	2	4	2	4	3	4	2	6

Table 1.8. Amounts of divisors (b)

Number	13	14	15	16	17	18	19	20	21	22	23
N of divisors	2	4	4	5	2	6	2	6	4	4	1

Our aim is to find the law regulating the numbers in the second row of the table. In other words, for any given natural number n, we want to find out the number of its divisors.

This task is not elementary. In order to get closer to the solution, we need to learn which of the properties of the number n have a decisive impact on the number of its divisors.

It is well known, that prime numbers serve as indivisible building blocks, from which all natural numbers can be derived using multiplication. This essential property of natural numbers is called the fundamental theorem of arithmetic.

Theorem 1.1 (The fundamental theorem of arithmetic)**.** *Every integer number greater than 1 can be represented as a product of prime numbers. This representation is unique up to the order of factors.*

This theorem requires clarification and explanation, which we provide below.

First, the theorem says about representations of all natural numbers greater than 1 as products of prime numbers. The statement of the theorem can be illustrated by many examples, such as $6 = 2 \cdot 3$, $8 = 2 \cdot 2 \cdot 2$, $12 = 2 \cdot 2 \cdot 3$, $18 = 2 \cdot 3 \cdot 3$, $70 = 2 \cdot 5 \cdot 7$ etc. But what about prime numbers? On the one hand, the theorem applies to them as well. On the other hand, no prime number can be expressed as a product of two or more prime numbers, because the only two divisors of any prime number are 1 and the number itself. Thus, the first statement of the theorem should be revised: every non-prime integer greater than 1 can be represented as a product of prime numbers. However, there is another opportunity. Let's assume that a prime number is a product consisting of only one factor. Then the first part of the theorem is true and requires no corrections.

The representation of a natural number as a product of prime numbers is called the decomposition of a number into prime factors (or the prime factorization (decomposition) of a number). Here is the decomposition of 120 into prime factors:

$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5.$$

Of course, one can change the order of factors in the right-hand part getting other decompositions of 120, such as $120 = 3 \cdot 2 \cdot 2 \cdot 5 \cdot 2$, $120 = 2 \cdot 5 \cdot 2 \cdot 3 \cdot 2$, etc., (incidentally, count

how many ways are there to arrange the product of five factors: $3, 5$ and three $2's$. You can apply the rule of product). However, disregarding the order of the factors we still have the same set of factors: three $2's$, 3, and 5. There is no other way to decompose 120 into prime factors. The second part of the theorem is about this property. The second part of the theorem does not make sense without the first one, but it does not provide less value. The facts that every natural number (except 1) can be decomposed into a product of prime numbers, and that such decomposition is unique (up to the order of factors), are equally essential.

Let $p_1, p_2, ..., p_5$ be different prime factors, into which the number n is decomposed, and $k_1, k_2, ..., k_5$ be numbers defining how many times the corresponding factors repeat in the decomposition. Assume, $p_1 < p_2 < p_3 < \cdots < p_5$. Then

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \cdots p_s^{k_s}$$

(instead of the product of the same factors p_i, we write the corresponding power of this number). This is one of many possible representations of the decomposition of the number n into primes. Because of the fact that the primes in this representation are sorted in increasing order, this representation is unequivocal. We call the standard (or canonic) decomposition of the number n into prime factors. Here are some examples of standard decompositions:

$$\begin{array}{ll} 48 = 2^4 \cdot 3; & 150 = 2 \cdot 3 \cdot 5^2; \\ 450 = 2 \cdot 3^2 \cdot 5^2; & 605 = 5 \cdot 11^2; \\ 1024 = 2^{10}; & 4563 = 3^3 \cdot 13^2. \end{array}$$

Following this preliminary work, which actually consisted of recalling several well-known facts from the course of arithmetic, we can strictly formulate the statement of the problem.

Let the canonical decomposition of the number n into prime factors be as follows:

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}.$$

How many natural divisors does the number n have?

If a is a natural divisor of n, then $n = a \cdot b$, and b is also a natural number. Now, if we decompose a and b into prime factors and input the derived expressions into the equality $n = a \cdot b$, then we get the decomposition of n into prime factors. Hence,

$$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s},$$
$$b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_s^{\beta_s}.$$

and $\alpha_1 + \beta_1 = k_1$, $\alpha_2 + \beta_2 = k_2$, ..., $\alpha_s + \beta_s = k_s$. Additionally, $\alpha_i \geq 0$ $(i = 1, ..., s)$ and $\beta_i \geq 0$ $(i = 1, ..., s)$, as a and b are integers. Therefore, if a is a divisor of n, then

$$a = p_1^{\alpha_1} p_2^{\alpha_2} ... p_S^{\alpha_S} \tag{1.2}$$

and $0 \leq \alpha_1 \leq k_1$, $0 \leq \alpha_2 \leq k_2$, ..., $0 \leq \alpha_s \leq k_s$.

On the other hand, provided the required inequalities for the powers α_i are fulfilled, every number a, constructed by formula (1.2), is a divisor of n, as

$$b = p_1^{k_1 - \alpha_1} \cdot p_2^{k_2 - \alpha_2} \cdots p_s^{k_s - \alpha_s}$$

is integer (because $k_i - \alpha_i \geq 0$ for $i = 1, 2, ..., s$) and

$$a \cdot b = n.$$

This fundamental result provides the answer to the question of the problem. It remains to solve it using the clearly combinatorial methodology.

We have learned that any divisor of n can be given by

$$p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_s^{\alpha_s}, \tag{1.3}$$

where $0 \leq \alpha_1 \leq k_1$, $0 \leq \alpha_2 \leq k_2$, $0 \leq \alpha_3 \leq k_3$, ..., $0 \leq \alpha_s \leq k_s$. On the other hand, if these inequalities are fulfilled for $\alpha_1, \alpha_2, ..., \alpha_s$, then expression (1.3) transforms into the divisor of n. Note that if two sets of powers in the product (1.3), e,g

$$\alpha_1', \alpha_2', \alpha_3', \ldots, \alpha_s'$$

and

$$\alpha_1'', \alpha_2'', \alpha_3'', \ldots, \alpha_s'',$$

differ from each other at least in one position ($\alpha_i' \neq \alpha_i''$ for some i), then

$$p_1^{\alpha_1'} \cdot p_2^{\alpha_2'} \cdots p_s^{\alpha_s'} \neq p_1^{\alpha_1''} \cdot p_2^{\alpha_2''} \cdots p_s^{\alpha_s''}.$$

This is because every natural number has a unique decomposition into prime factors.

Summing up the facts presented above, we come to a conclusion.

The number $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$ has the amount of divisors, which is equal to the amount of expressions (1.3), where α_1 is some integer from 0 to k_1 inclusive, α_2 is some integer from 0 to k_2 inclusive, and so on. Finally, α_s is some integer from 0 to k_s inclusive. Hence, there is $k_1 + 1$ possibilities for α_1, there is $k_2 + 1$ possibilities for α_2, and so on. Finally, there $k_s + 1$ possibilities for α_s. In addition, the values for $\alpha_1, \alpha_2, \ldots, \alpha_s$ can be chosen independently. In other words, the values of $\alpha_1, \alpha_2, \ldots, \alpha_s$ can be randomly combined with each other. As we already know, this is the sign of applicability of the rule of product. Hence, the overall amount of sequences

Table 1.9. Sequences of α_k

α_1	α_2	α_3	$\ldots$	α_s

equals to the product

$$(k_1 + 1) \cdot (k_2 + 1) \cdot (k_3 + 1) \cdots (k_s + 1).$$

Thus, this is the number of natural divisors of n.

For instance, the number 675 has the following canonical decomposition into prime factors:

$$675 = 3^3 \cdot 5^2,$$

and therefore, has $(3+1) \cdot (2+1) = 12$ natural divisors. Its divisors are numbers of the form

$$3^{\alpha} \cdot 5^{\beta},$$

where α obtains values from the set $0,1,2,3$, and the domain of β is $0,1,2$. In particular, the divisor corresponding to the values $\alpha = 0$ and $\beta = 0$ is 1, for $\alpha = 2$, $\beta = 1$, the divisor is 45 and so on.

Problems

Problem 1.44. *How many four-digit numbers can be written using odd digits only? Calculate the sum of all these numbers.*

Solution. Applying the rule of product, we easily get the answer to the first part of the problem: there are 5^4 numbers. The second task can be solved as follows. First, answer the question: how many numbers are there among those mentioned in the problem, which in the place of ones have 1? The answer is (by virtue of the aforementioned rule of product) 5^3.

Next, how many numbers are there, having 3 in the place of ones? Similarly, 5^3. By analogy, there are 5^3 numbers with 5 in the place of ones, 5^3 numbers having 7 in this place, and 5^3 numbers with 9. Therefore, the sum of digits in the place of ones in the numbers of interest is

$$5^3 \cdot 1 + 5^3 \cdot 3 + 5^3 \cdot 5 + 5^3 \cdot 7 + 5^3 \cdot 9 =$$
$$= 5^3 \cdot (1+3+5+7+9) = 5^3 \cdot 25 = 5^5.$$

Moreover, the above considerations, apply to the rest of the three digits of the numbers without any changes. Thus, we conclude that the sum of all numbers is

$$5^5 + 5^5 \cdot 10 + 5^5 \cdot 10^2 + 5^5 \cdot 10^3 = 5^5 \cdot 1111.$$

Based on a completely different idea, there is another elegant solution to the second part of the problem he solution to the second task. Imagine that all 5^4 numbers are written in a column following random (no matter what) rule (for example, in ascending order). Corresponding to each number a from the first column, we construct the number b in the second column such that the sum of digits in every place of numbers a and b is 10. For instance, if the first column is arranged in ascending order, then the columns look as follows:

$$\left|\begin{array}{cc} 1111 & 9999 \\ 1113 & 9997 \\ 1115 & 9995 \\ 1117 & 9993 \\ 1119 & 9991 \\ 1131 & 9979 \\ \dots & \dots \\ 9997 & 1113 \\ 9999 & 1111 \end{array}\right| .$$

There are the same numbers in the second column, as in the first, but in a different order. This follows from the properties: 1) if the digit γ is odd, then the digit $10-\gamma$ is odd; 2) if numbers from the first column differ, then the corresponding numbers from the second column also differ. Taking into account these facts, it is now straightforward to sum up the

numbers of the first column. The sum of any two numbers standing in the same is 11110. Therefore, the sum of all numbers of both columns is $11110 \cdot 5^4 = 2 \cdot 1111 \cdot 5^5$, and the sum of numbers in one column is half that amount, that is $1111 \cdot 5^5$.

Problem 1.45. *How many four-digit numbers can be written using no odd digits? Calculate the sum of all these numbers.*

Answer. $4 \cdot 5^3$; 2722000

Problem 1.46. *How many different four-digit numbers are there consisting of the digits* 2, 4, 6 *and* 8? *Calculate the sum of all these numbers.*

Answer. 256; 1422080.

Problem 1.47. *How many different four-digit numbers can be written using odd digits only, with no digit being used two or more times in a row? Calculate the sum of these figures using at least two ways.*

Answer. 320; 1777600.

Problem 1.48. *How many four-digit numbers are there, which are divisible by 3 and consist of the digits* 0, 1, 2, 3, 4, 5, 6 *only?*

Answer. $2 \cdot 7^3$.

Hint. *You can apply the rule of product moving through the digits of the wanted numbers from left to right (and only in this way).*

Problem 1.49. *How many four-digit numbers are there, which are divisible by* 3 *and consist of the digits* 0, 1, 2, 3, 4, 5 *only?*

Answer. 360.

Hint. *You can apply the rule of product moving through the digits of the wanted numbers from left to right (and only in this way).*

Problem 1.50. *How many five-digit numbers are there, which are divisible by 4 and consist of even digits only?*

Answer. 1500.

Problem 1.51. *How many four-digits can be written using at least one zero?*

Answer. $9 \cdot 10^3 - 9^4$.

Problem 1.52. *How many four-digit numbers are there, each digit of which is the same as the previous one or exceeds it by* 1?

Answer. 60.

Sketch of Solution. If a number begins with the digit 1, or 2, or 3, or 4, or 5, or 6, then moving to each subsequent digit, we can use both provided options: the next digit may be either the same ash the previous one, or greater than it by 1. Therefore, there are $6 \cdot 2 \cdot 2 \cdot 2 = 48$ such numbers.

If a number begins with 7, then one out of eight options for the values of the next three digits is not applicable, – the one, where there are no same digits. So there are only 7 such numbers.

If the first digit is 8, then the rest of its digits can be only 8 and 9. There are 4 such numbers.

Finally, there is only one digit with the first digit being 9.

Hence, we arrive to the answer:

$$48+7+4+1=60.$$

Problem 1.53. *How many four-digit numbers are there, any two adjacent digits of which does not differ by more than* 1?

Answer. 217.

Solution. If a number begins with the digit 1, or 2, or 3, or 4, or 5, or 6, then moving to each subsequent digit, we can use both provided options: the next digit may be either the same as the previous one, or greater than it by 1. Therefore, there are $6 \cdot 2 \cdot 2 \cdot 2 = 48$ such numbers.

If a number begins with 7, then one out of eight options for the values of the next three digits is not applicable, – the one, where there are no same digits. If such a number starts with one of the numbers 3, 4, 5 or 6, then on each place there are three options for a digit to choose: the same as the previous one, or greater/lesser by 1 than the previous one. Therefore, there are $4 \cdot 3 \cdot 3 \cdot 3 = 108$ numbers starting with these four digits.

When the initial digit is 2, then it's no longer possible to behave freely with its subsequent digits. For example, if we reduce each subsequent digit by 1 (compared to the previous one), we will get -1 in the fourth (last) position and this is not a digit. It is clear that this option is the only unacceptable among 27, because it violates the rules of the game, destroying the structure of the number. Therefore, there are 26 numbers with the first digit 2. Similarly, there are 26 numbers beginning with 7. In this case, the option where each following digit is greater than the previous one is unacceptable.

Now, let the first digit be 1. Each next digit must either be the same as the previous, or differ from it by 1. It is crucial that the implementation of these options does not lead to the appearance of numbers less than 0 or greater than 9. The procedure for constructing a number with a given initial digit can be described by a code composed of four numbers: the first being the given digit, and each of the others being one of the numbers 1, 0 or -1 depending on what needs to be done to the previous digit of the number to get the next. For example, $(1;1;-1;0)$ is the code of the number 1211, and $(1;-1;-1;0)$ is the code of the "fictitious" number $10(-1)(-1)$, as an application of the procedure denoted by the third component of the code, leads to the number, which is less than the lowest digit. There are only 5 codes with the first component being 1, which does not correspond to existing numbers: $(1;-1;-1;-1)$, $(1;-1;-1;0)$, $(1;-1;-1;1)$, $(1;-1;0;-1)$ and $(1;0;-1;-1)$. Hence, there are 22 numbers, beginning with 1.

There are 22 numbers starting with 8 as well. This fact is a consequence of a peculiar symmetry: digits 1 and 8 rank second from opposite ends in the hierarchy of digits. The rule of forming the numbers expressed by the codes retains this symmetry. Here are five codes that correspond to fictitious numbers with an initial digit of $8:(8;1;1;1)$, $(8;1;1;0)$, $(8;1;1;-1)$, $(8;1;0;1)$ and $(8;0;1;1)$. Compare these codes with the codes in the case where the first digit is 2, to see the symmetry mentioned above.

Finally, we provide the codes for fictitious numbers, beginning with $9:(9;1;\alpha;\alpha)$, $(9;0;1;\alpha)$, $(9;0;0;1)$ and $(9;-1;1;1)$. The letter α denotes any of three numbers: $-1, 0, 1$. Overall, there are 14 such codes. Hence, there are 13 numbers starting with 9.

We have found the answer to the problem. There are

$$108+2\cdot 26+2\cdot 22+13=217$$

wanted numbers.

Problem 1.54. *How many eight-digit numbers can be written using only two digits: 1) 0 and 1? 2) 1 and 2?*

Answer. *1)* 2^7*; 2)* 2^8.

Problem 1.55. *How many four-digit numbers are there, with a sum of any two adjacent digits being: a) odd? b) even?*

Answer. *a)* 1125; *b)* 1125.

Problem 1.56. *How many four-digit numbers are there, with a product of any two adjacent digits being: a) odd? b) even?*

Answer. *a)* 5^4*; b)* 3750.

Hint. a) This is possible only if all the digits are odd. b) This is possible only if a number does not have odd adjacent digits.

Problem 1.57. *How many four-digit numbers are there, with the sum of its digits being: a) even? b) divisible by* 5*? c) divisible by* 10?

Answer. *a)* 4500; *b)* 1800; *c)* 900.

Problem 1.58. *How many four-digit numbers have: a) one zero? b) exactly two zeros?*

Answer. *a)* $3\cdot 9^3$*; b)* $3\cdot 9^2$.

Problem 1.59. *How many five-digit numbers are there, in which the same digits (if any) are separated from each other by at least two positions?*

Answer. $9^2\cdot 8^3$.

Problem 1.60. *1) How many seven-digit numbers can be written using the digits 1, 2 and 3 only? 2) How many of them have all these digits?*

Answer. *1)* 2187; *2)* 1806.

Problem 1.61. *A five-digit number comprise odd and even digits, and the former are followed by the latter (for example:* 17802, 92448, 93772 *etc.). How many such numbers are there?*

Answer. $4\cdot 5^5$.

Problem 1.62. *A five-digit number comprises even and odd digits, and the former digits are followed by the latter digits. How many such numbers are there?*

Answer. $4^2\cdot 5^4$.

Problem 1.63. *A five-digit number begins with an odd digit and ends with an even one. How many such numbers are there?*

Answer. $5^2\cdot 10^3$.

Problem 1.64. *A five-digit number begins with an even digit and ends with an odd one. How many such numbers are there?*

Answer. $4\cdot 5\cdot 10^3$.

Problem 1.65. *A five-digit number is divisible by* 3. *The numbers, which one can get by crossing out one, two, three, or four last digits from the initial five-digit number, are also divisible by* 3. *How many such five-digit numbers exist? Calculate the sum of all these numbers.*

<u>*Answer.*</u> $3 \cdot 4^4$; 49919616.

Problem 1.66. *A seven-digit number is divisible by* 3. *The numbers, which one can get by crossing out one, two, three, or four last digits from the initial five-digit number, are also divisible by* 3. *How many such five-digit numbers exist?*

<u>*Answer.*</u> $300 \cdot 4^4$.

Problem 1.67. *A six-digit number has the following properties: firstly, it is divisible by* 4; *secondly, the four-digit number, which one gets by crossing out the last two digits of the initial six-digit number, is also divisible by* 4; *thirdly, crossing out the last two digits of this four-digit number, one gets a two-digit number still divisible by 4. Provide several examples of such six-digit numbers. How many such numbers exist?*

<u>*Answer.*</u> *There are* 13750 *such numbers.*

<u>*Hint.*</u> *Recall the criterion of divisibility by* 4.

Problem 1.68. *A six-digit number is divisible by* 4. *The numbers, which one can get by crossing out one, two, or three from the initial six-digit number, are also divisible by* 4. *Provide* $3-4$ *examples of such six-digit numbers. How many such six-digit numbers exist?*

<u>*Answer.*</u> 6075.

Problem 1.69. *A six-digit number is divisible by* 5. *The numbers, which one can get by crossing out one, two, or three from the initial six-digit number, are also divisible by* 5. *Provide several examples of such six-digit numbers. How many such six-digit numbers exist? Calculate the sum of all these numbers.*

<u>*Answer.*</u> 1440; 788799600.

Problem 1.70. *We write down five-digit numbers, strictly adhering to the following rules. The first digit should be divisible by* 3. *Each next digit should be divisible by* 4 *if the previous one is divisible by* 3 *and, conversely, it should be divisible by* 3 *if the previous one is divisible by 4. How many such five-digit numbers exist?*

<u>*Answer.*</u> 171.

<u>*Hint.*</u> *Count the numbers having at least one zero and not having any zeroes separately.*

Problem 1.71. *There are n "red" points on the straight line l_1 and m "blue" points on the straight line l_2, which is parallel to l_1. How many segments can be drawn with endpoints of different colors?*

<u>*Answer.*</u> *nm.*

Problem 1.72. *There are k "red" points on the first side of a triangle, m "blue" points on the second and n "green" points on the third. How many triangles are there with vertices of different colors?*

<u>*Answer.*</u> *kmn.*

Problem 1.73. *How many six-digit numbers are there, which are divisible by* 9 *and does not contain zero?*

Answer. 9^5.

Problem 1.74. *1) How many four-digit numbers divisible by 25 can be written using the digits* 2, 3, 4, 5, 6, 7, 8 *only?*

2) How many of them have none of their digits repeating?

Answer. *1)* 98; *2)* 40.

Problem 1.75. *How many five-digit numbers are there, in which not all digits are different?*

Answer. $9 \cdot 10^4 - 9^2 \cdot 8 \cdot 7 \cdot 6$.

Problem 1.76. *How many five-digit numbers have at least one digit 1?*

Answer. $9 \cdot 10^4 - 8 \cdot 9^4$.

Problem 1.77. *How many ways are there to place eight rooks of the same color on an empty chessboard so that they do not attack each other?*

Answer. $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Hint. The product of all positive integers less than or equal to n is denoted by the symbol $n!$. Thus, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. For example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$, $6! = 5! \cdot 6 = 720$ etc. Hence, the answer to the problem can be expressed as $8!$.

Problem 1.78. *In how many ways can a soccer team of 11 players be lined up in a column?*

Answer. $11!$

Problem 1.79. *How many ways are there to place eight rooks of eight different colors (fairytale-like chess) on an empty chessboard so that they do not attack each other?*

Answer. $(8!)^2$.

Problem 1.80. *The sides AB and BC of the triangle ABC are split into n parts each. Splitting points on the side AB are connected using line segments with the vertex C, and splitting points on the side BC, – with the vertex A. 1) How many points do the drawn line segments intersect inside the triangle? 2) How many parts do they split the triangle into?*

Answer. *1)* $(n-1)^2$; *2)* n^2.

Problem 1.81. *(Trajectories in a circle) There are n points on a circle* $(n \geq 2)$. *For our convenience, we call them "base" points further. We choose one of them to be an initial point and denote it A.*

I. Calculate the amount of polygonal chains with the following properties:

1. *One end of a polygonal chain is the point A, and the other one is any other base point.*
2. *Segments of a polygonal chain are chords connecting the base points.*
3. *Every base point, except for the ending points of a polygonal chain, is a vertex separating its line segments.*
4. *There are no intersections of line segments of a polygonal chain inside the circle (a simple polygonal chain).*

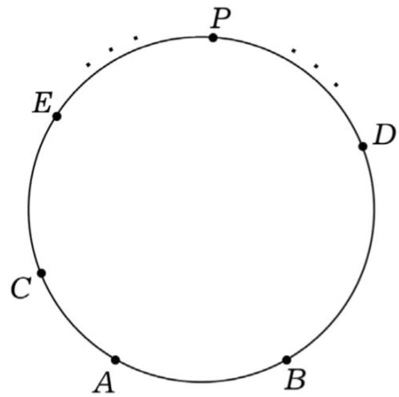

Figure 1.7. Trajectories in a circle.

II. How many polygonal chains satisfying properties 2), 3) and 4) are there, having any two different base points as endpoints?

Answer. *I.* 2^{n-2}*; II.* 2^{n-3}*.* n*.*

Sketch of Solution. Let B and C be base points adjacent to the initial base point A. This means that there is no other base points except the point A on one of the curves, which the points B and C split the circle into. Any polygonal chain satisfying the conditions of the problem can begin with the intervals AB or AC only because any other interval (say, AD, AE, AF etc.) would split the circle into two segments, each having "unvisited" base points on its bounding arc (e.g. the point B would be in the one segment and the point C in the other). According to the statement of the problem, our chain will visit all the base points eventually, and hence, will intersect with the first chord.

Therefore, standing in the point A and deciding which chord to follow, or which means the same, which base point to visit first, we actually have only two options. The first point to visit is the point B, or the point C. None of these options has any advantages over the other. Whichever of these points we step into from the point A, it does not affect the future prospects of the construction of the polygonal chain. Now, the point has been left behind and is not of our interest anymore. The point we are currently standing in (that is, B or C) can be considered initial. The only difference is that we have $n-2$ points ahead now instead of $n-1$, as it has been in the beginning. Hence, after the first step, we are in the initial setting, except for the number of base points has reduced by 1. There is the same choice for the second step, as it has been for the first: our route will lead to one of the adjacent points (not taking into account the point A). This is a good time to use the "and so on" argument, as after each next step (after each new line segment of the polygonal chain is drawn), the previous situation is restored, provided there are two or more points to visit.

Now, we are getting close to a solution. The required polygonal chain has $n-1$ line segments. Starting from the point A and moving from one base point to another, at each step we need to choose from two options: move to the adjacent point on the left, or on the right. Hence, we have two options for the first line segment, two for the second, two for the third, and so on. The only exception is the last, $(n-1)$-th line segment. When we

arrive at the $(n-1)$-th point, we have only one unvisited base point left. There is no choice anymore. Our route is coming to its end with one more obligatory line segment, which leads to the final base point. There is only one option for the final, $(n-1)$-th line segment. Therefore, there are two options for each line segment from the first up to the $(n-2)$-th and one option for the $(n-1)$-th line segment. Moreover, numbers 2 and 1, defining the freedom of choice on every stage of construction of the polygonal chain, do not depend on the previous choices. This completely resembles the combinatorial rule of product, hence, the amount of possible polygonal chains is

$$\underbrace{2\cdot 2\cdot 2\cdot \ldots \cdot 2}_{n-2\,times}\cdot 1 = 2^{n-2}.$$

II. According to the first part of the problem, there are 2^{n-2} polygonal chains with the fixed starting point A. The role of A can be played by any base point now, and there are n of them. However, the product $2^{n-2}\cdot n$ is not the wanted number, as any chain has two ends. Multiplying 2^{n-2} by n, we account for each polygonal chain twice. Hence, the correct answer is $2^{n-3}\cdot n$.

Problem 1.82. *The canonical decomposition of the number a into prime factors is as follows:*

$$a = 2^m \cdot p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$$

(p_1, p_2, .., p_s - different prime odd numbers).

How many even divisors does a have?

<u>*Answer.*</u> $m\cdot(k_1+1)\cdot\ldots\cdot(k_s+1)$.

Problem 1.83. *If one chooses a rectangular coordinate system in 3-space and places a cube in it, so that its three edges lie on the coordinate axes, and the whole cube is in the first octant, then the eight vertices of the cube have the following coordinates:* $(0;0;0)$, $(1;0;0)$, $(1;1;0)$, $(0;1;0)$, $(0;0;1)$, $(1;0;1)$, $(1;1;1)$ *and* $(0;1;1)$ *(Fig. 1.8).*

A cube has 12 *edges. Each of them is bounded by two points, which are the vertices of a cube. However, any two vertices do not necessarily bound an edge (that is, represent the ends of some edge). How can we distinguish those pairs of vertices, which are the ends of some edge? Having thoroughly studied the figure, one can find the answer. The ends of the same edge are those vertices, the coordinates of which differ in <u>one</u> coordinate only. It does not matter which coordinate differs. Any edge can be defined by defining (in any order) the vertices, which bound it. For instance,* $[(0;0;0),(1;0;0)]$ *is the edge of the cube lying on the x-axis, and* $[(1;0;1),(1;1;1)]$ *is the one, which is parallel to the y-axis and lying outside of all the coordinate planes.*

Similarly, it is easy to find the condition, under which two vertices of the cube are the ends of its diagonal. The points, bounding the diagonals of the cube, differ in <u>every</u> coordinate. They have different first, second, and third coordinates. One of the diagonals is drawn on the figure. Its ends are the points (0; 0; 1) and (1; 1; 0).

The notion of vertices, edges, and diagonals of a three-dimensional cube can be formally arithmetically generalized for the case of n dimensions.

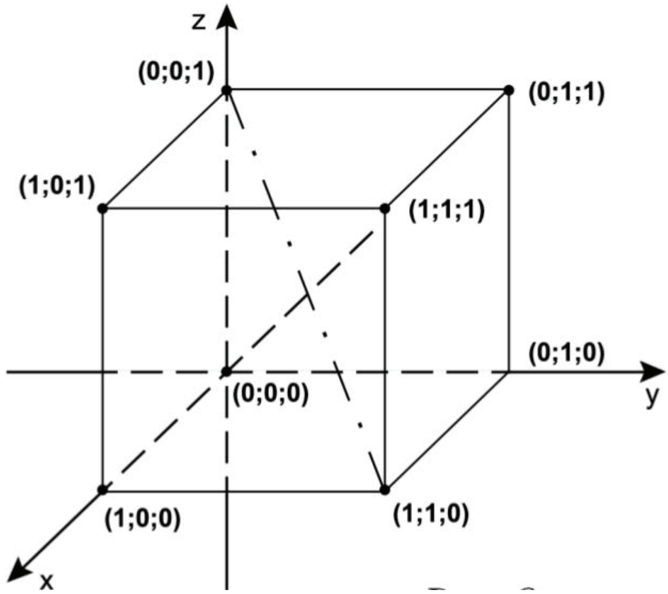

Figure 1.8. Cube.

Fix any natural number $n \geq 3$. *n-dimensional points* $(a_1; a_2; ...; a_n)$, *the coordinates of which* a_i *can be either 0, or 1, are called vertices of the n-dimensional cube (having edges of length 1). We assume, that the previous statement fully defines all the vertices of the cube, that is the n-dimensional cube does not have any other vertices.*

The pair $[A, B]$, *whereA and B denote vertices of the (n-dimensional) cube is its edge if these vertices differ in one coordinate only. Naturally, we consider* $[A, B]$ *and* $[B, A]$ *to be the same edge.Finally, the pair of vertices* $[A, B]$ *of the (n-dimensional) cube is its diagonal if the vertices A and B differ in every coordinate. Again,* $[A, B]$ *and* $[B, A]$ *denote the same diagonal.*

In order to make things completely clear, we provide examples for the case of $n = 5$, *which illustrate the above notions.*

The vertices of five-dimensional cube are:

$(1;1;0;0;1), (0;1;0;0;1), (0;0;0;0;1)$ *etc.*

Some of its edges are given by

$[(1;1;0;0;1), (0;1;0;0;1)], [(0;0;0;0;0), (0;0;0;0;1)],$
$[(0;0;0;0;0), (0;1;0;0;0)], [(0;1;0;1;1), (0;1;1;1;1)].$

Finally, its diagonals are given as follows:

$[(0;0;1;1;0), (1;1;0;0;1)], [(1;1;0;0;0), (1;1;0;0;1)].$

Recall that two vertices of a (regular) three-dimensional cube are called adjacent if they are connected by an edge, and they are called opposite otherwise. We can use the same notions in the case of the n-dimensional cube. Adjacent vertices are those, differing in one coordinate, and the opposite ones are those, differing in each of n coordinates.

Now, we suggest the reader to answer several questions.

I. How many vertices does the n-dimensional cube have?

II. Let M be one of the vertices of an n-dimensional cube. How many vertices of the cube are adjacent to M? (The question can be formulated in another way: how many edges start from the vertex M?)

III. How many edges does the n-dimensional cube have?

IV. How many diagonals does the n-dimensional cube have?

V. A spider can run along edges of the n-dimensional cube. It has to get from the vertex $(0;0;0;\ldots;0)$ *to the vertex* $(1;1;1;\ldots;1)$ *(that is, to the opposite vertex). The spider intends to reach its aim using the shortest possible way. What is the length of such a way? How many different shortest ways are there for the spider? (in other words, how many different shortest ways are there to get from the vertex* $(0;0;0;\ldots;0)$ *of the n-dimensional cube to the opposite vertex?)*

Answer. *I.* 2^n*; II. n; III.* $2^{n-1}\cdot n$*; IV.* 2^{n-1}*; V. n;* $n!$ *(the symbol* $n!$ *– n-factorial – denotes the product of all natural numbers from 1 to n:* $n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n$*).*

Problem 1.84. *How many shortest different ways are there along edges of the seven-dimensional cube from the vertex* $(1;0;1;1;0;0;1)$ *to the vertex* $(0;0;1;0;1;1;1)$*? (Refer to Problem 1.83 for the terminology).*

Answer. 24.

Problem 1.85.

1. *Horizontal and vertical strips split a rectangle into small squares. A horizontal strip contains 5 squares, and the vertical one contains* 4.

2. *How many ways are there to fill in the squares with the numbers* 1 *and* -1 *putting one number into each square?*

3. *The numbers* 1 *and* -1 *have to be placed into the squares in such a way that the product of numbers on any horizontal strip equals* 1. *How many ways are there to fill the numbers 1 and* -1 *into the rectangle under this restriction?*

4. *The numbers* 1 *and* -1 *have to be placed into the squares in such a way that the product of numbers on any horizontal or vertical strip equals* 1. *How many ways are there to fill the numbers 1 and* -1 *into the rectangle under this restriction?*

5. *How many ways are there to fill the numbers* 1 *and* -1 *into the squares so that the product of all the* 20 *numbers is* 1?

6. *The numbers* 1 *and* -1 *have to be placed into the squares in such a way that any horizontal strip contains only one number* 1. *How many ways are there to fill the numbers* 1 *and* -1 *into the rectangle under this restriction?*

7. *The numbers* 1 *and* -1 *have to be placed into the squares in such a way that any horizontal strip contains only one number* 1 *and these ones should be placed in different vertical strips. How many ways are there to fill the numbers 1 and* -1 *into the rectangle under this restriction?*

Answer. *1)* 2^{20}*; 2)* 2^{16}*; 3)* 2^{12}*; 4)* 2^{19}*; 5)* 4^5*; 6) 5!*

Solution of Part 3. This problem is the most difficult and at the same time, the most interesting among the six proposed. We will try to act straightforward and fit the combinatorial rule of product to its solution. Of course, we can not fill the numbers 1 and -1

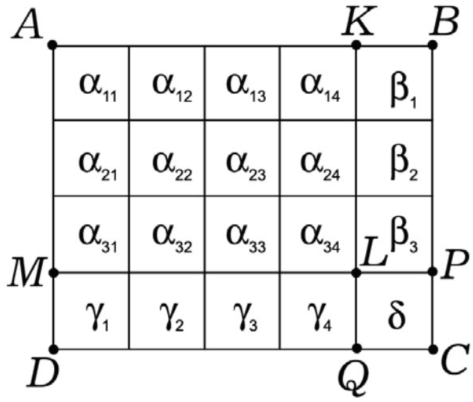

Figure 1.9. Solution to part 3.

in the squares randomly and independently from each other, because the problem imposes extremely strict restrictions on the resulting array of numbers. However, we can attempt to find some of the squares in which the numbers 1 and -1 can be put randomly, hoping to "save the day", that is, to satisfy the stated requirements, at the expense of the remaining squares. It seems logical to choose the rectangle *AKLM* Fig. 1.9 as a "random field" for putting 1 and -1 randomly. *AKLM* is derived from the original rectangle by discarding the bottom horizontal strip and the right vertical strip. Why we choose this rectangle? Because properly selecting numbers that we put into squares of rectangles *KBPL* and *MLQD*, we will be able to achieve the desired product in three upper horizontal and four left vertical strips. After that, nothing will depend on us. Which number to fill in the square *LPCQ*: 1 or -1? The answer will be pre-determined by the numbers in the rectangle *KBPL.* On the other hand, the numbers in the rectangle *MLQD* will also dictate the answer to the question. If the two above sources provide the same answer, then the problem is solved. Otherwise, our choice of the "random field" has been incorrect and we have to search it elsewhere or even abandon this approach.

Fortunately, the rectangles *KBPL* and *MLQD* always provide agreed directions concerning the number to be filled into the square *LPCQ*. Consider this fact in more detail. So let us have randomly inserted into the numbers 1 and -1 into the squares inside the rectangle *AKLM*. On Fig. 9 these numbers are denoted by the letter α with two indices below. Now, fill the numbers β_1, β_2 and β_3 into the squares of the rectangle *KBPL* following the rule

$$\beta_1 = \alpha_{11}\cdot\alpha_{12}\cdot\alpha_{13}\cdot\alpha_{14},\ \beta_2 = \alpha_{21}\cdot\alpha_{22}\cdot\alpha_{23}\cdot\alpha_{24},\ \beta_3 = \alpha_{31}\cdot\alpha_{32}\cdot\alpha_{33}\cdot\alpha_{34},$$

and the numbers $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ into the squares of the rectangle *MLQD* by the rule

$$\gamma_1 = \alpha_{11}\cdot\alpha_{21}\cdot\alpha_{31},\ \gamma_2 = \alpha_{12}\cdot\alpha_{22}\cdot\alpha_{32},\ \gamma_3 = \alpha_{13}\cdot\alpha_{23}\cdot\alpha_{33},\ \gamma_4 = \alpha_{14}\cdot\alpha_{24}\cdot\alpha_{34}.$$

These rules guarantee that the product of numbers of any of the top three horizontal and any of the left four vertical strips of the rectangle *ABCD* equals to 1. For instance, for the top horizontal strip, we have:

$$\alpha_{11}\cdot\alpha_{12}\cdot\alpha_{13}\cdot\alpha_{14}\cdot\beta_1 = (\alpha_{11}\cdot\alpha_{12}\cdot\alpha_{13}\cdot\alpha_{14})^2 = 1.$$

The same happens in the other cases.

The number in the square *LPCQ* has to satisfy two purposes now: it must ensure that the products of numbers in the bottom horizontal and the right vertical strip equal to 1. Whether it is possible or not, depends on the numbers β_1, β_2, β_3 and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ only. The product of numbers in the bottom strip equals to 1, if $\delta = \gamma_1\gamma_2\gamma_3\gamma_4$, and the product of numbers in the right strip equals 1, if $\delta = \beta_1\beta_2\beta_3$. Taking into account the way the numbers β_1 and γ_k are expressed in terms of γ_{pq}, it is straightforward to ensure that

$$\beta_1\beta_2\beta_3 = \gamma_1\gamma_2\gamma_3\gamma_4$$

(both these products are equal to the product of all the numbers γ_{pq} from the rectangle *AKLM*). Here is the deal. That is why a random set of numbers from the rectangle *AKLM* can be expanded (and in addition, in a unique way) on the whole rectangle *ABCD*, keeping the constraints satisfied. Hence, we conclude that one can create as many different rectangular numerical 5×4 tables as required by the task, as there are rectangular 4×3 tables of the numbers 1 and -1.

Problem 1.86. *Formulate the above problem in the "general" case, when a $m \times n$ rectangle is split by line segments parallel to its sides into m horizontal and n vertical strips, that is, into mn small squares. Pose all the questions from the previous problem with respect to this rectangle and find answers to them.*

Problem 1.87. *Andrew has 5 red, 5 blue and 5 green cubes differing only by their color. He constructs five-store columns from them. After he builds one, he destroys it and begins constructing a new one. Each following column should differ from all the previous. How many columns can Andrew build?*

Answer. 243.

Problem 1.88. *How many ways are there to place five books next to each other on a shelf?*

Answer. 120.

Problem 1.89. *In Fig. 1.10, there is a fragment of a rectangular coordinate system on the plane. Let us call the straight lines $x = k$ $(k = 0, \pm 1, ...)$ and $y = s$ $(s = 0, \pm 1, ...)$ the integer-valued coordinate lines. A spider sitting at the origin can run strictly along these lines and wants to get to the interval AB in the shortest possible way. Which distance does he need to cover? How many ways are there for him to choose a route?*

Answer. *The shortest route is of length 7. There are 2^7 routes of length 7.*

Solution. There is the same distance from the point O to all the integer (that is, with integer coordinates) points of the interval *AB*. In order to get them, one needs to move up or to the right along the coordinate lines. A step to the right (on the distance of 1) increases the abscissa of the point by 1 and a step-up increases the ordinate. The sum of coordinates of all the points of the interval *AB* is 7. Hence, in 7 steps the spider finds himself in one of them. In any of seven crossings, which the spider needs to pass, it can choose either to move along Ox or along Oy. Therefore, there are 2^7 different routes (one of them is outlined in the figure).

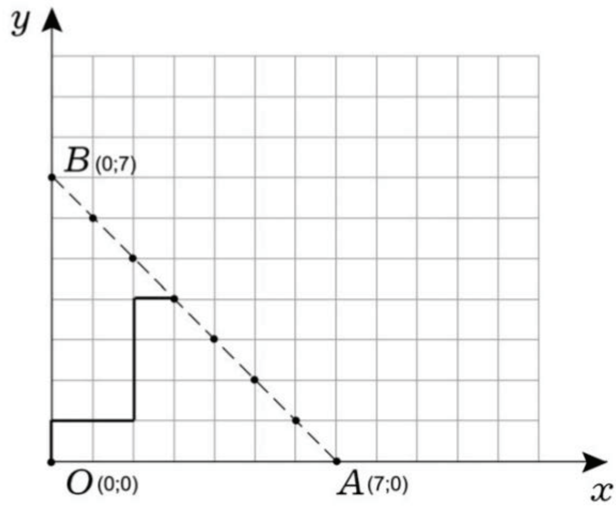

Figure 1.10. The shortest route.

Problem 1.90. *(Problem 1.89 continued). How many ways are there to choose the shortest route along the integer-valued coordinate lines from the point O to the interval AB, under the condition that this route should consist of:*

1. *one line segment?*
2. *two line segments?*
3. *three line segments?*

Answer. *1)* 2; *2)* 12; *3)* 30.

Problem 1.91. *Construct a generalization of problems 1.89 and 1.90, assuming that the interval AB connects the points* $A(n;0)$ *and* $B(0;n)$*. Find the answers to all five questions from problems 1.89and 1.90 in this case.*

Answer. n; 2^n; 2; $2(n-1)$; $(n-2)(n-1)$.

Problem 1.92. *Draw the line* $|x|+|y|=7$ *on the coordinate plane. How many shortest ways are there to reach this line from the point of origin* $O(0;0)$ *(along the integer-valued coordinate lines)?*

Answer. $4\cdot(2^n-1)$.

Problem 1.93. *The spatial grid is constructed with the integer-valued coordinate lines*
$\begin{cases} x=m \\ y=k \end{cases} (m,k\in Z)$; $\begin{cases} x=m \\ z=k \end{cases} (m,k\in Z)$ *and* $\begin{cases} y=m \\ z=k \end{cases} (m,k\in Z)$.

A spider can run along the lines of this grid (that is, along with its web). It stands in the point of origin and intends to reach the plane $x+y+z=n$ *(n is fixed natural number) in the shortest possible way.*

1. *Which distance does the spider have to cover?*

2. *How many points are there on the plane $x+y+z=n$, in which the spider can finish his route?*

3. *How many different routes are available for him?*

4. *Let the spider decide to follow the route, which completely lies in one plane (that is, the plane route). How many options does he have now?*

5. *How many ways to choose the shortest route would he have if he decided to follow a route consisting of:*

   a) two line segments?

   b) two line segments lying in the same plane?

   c) three line segments not lying in the same plane?

Answer. *1)* n; *2)* $\frac{1}{2}(n+1)(n+2)$; *3)* 3^n; *4)* $3\cdot(2^n-1)$; *5) a)* $6\cdot(n-1)$; *b)* $3(n-2)(n-1)$; *c)* $3(n-2)(n-1)$.

Problem 1.94. *There is an encrypting system, which transforms messages into sequences of five vowel (a, e, i, o, u) and five consonants (b, c, d, f, g). Each letter should appear in a transformed message only once. Here are examples of three different messages: abcdefgiou, aeioubcdfg, uoibcdaefg.*

1. *How many different encryptions can this system produce?*

2. *How many messages can begin with five consonants followed by five vowels?*

3. *How many encryptions consist of vowels alternating with consonants?*

4. *How many encryptions are there having all vowels grouped together?*

5. *How many encryptions begin with a vowel?*

6. *How many encryptions begin with a vowel and end up with a consonant?*

7. *How many encryptions begin and end up with a consonant?*

8. *How many encryptions have the adjacent letters "a" and "e"?*

9. *How many encryptions have two letters between "a" and "e"?*

10. *How many encryptions begin with three vowels followed by a consonant?*

Answer. *1)* $10!$ *(this denotes the product of all natural numbers from* 1 *to* 10 *inclusive; hence, there are* 3628800 *messages); 2)* $(5!)^2$; *3)* $2\cdot(5!)^2$; *4)* $5!\cdot 6!$; *5)* $5\cdot 9!$; *6)* $5\cdot 5\cdot 8!$; *7)* $5\cdot 4\cdot 8!$; *8)* $2\cdot 9!$; *9)* $2\cdot 7\cdot 8!$; *10)* $5\cdot 4\cdot 3\cdot 5\cdot 6!$.

Problem 1.95. *A draw of the round of* 16 *of a football cup is to be performed by the following rules. The 16 teams entering this round are split into two groups of* 8 *teams each according to their last year's rating. Balls with the names of the stronger teams are put into one bowl and the weaker ones are put into another. At the drawing ceremony, two balls are taken from separate bowls and the names of the competing teams are announced. How many ways are there for the drawing to finish: 1) for one team? 2) for all the teams?*
Answer. *1)* 8; *2)* $8!$.

Problem 1.96. *How many natural solutions are there to the system of equations*

$$\begin{cases} x+y+z=t=40, \\ x-y+z-t=32? \end{cases}$$

How many integer non-negative solutions exist?

Answer. 105; 185.

Preliminary Information. A solution to a system of equations with four variables x, y, z, t is the ordered set of numbers $(a; b; c; d)$, which posses the following property: replacing x for a, y for b, z for c, and t for d in the equations, we get correct numerical equalities. The numbers a, b, c and d are called the first, second, third and fourth components of the solution respectively. For instance, the quad $(36; 4; 0; 0)$ is one of many solutions to the system in question. A solution is called natural if all its components are natural numbers. A solution is called integer non-negative if all its components are natural numbers or zeros.

Problem 1.97. *How many natural solutions are there to the system of equations*

$$\begin{cases} 2x+y+z=52, \\ y-z=10? \end{cases}$$

How many integer non-negative solutions exist?

Answer. 20; 22.

Problem 1.98. *How many natural and integer non-negative solutions are there to the system of equations*

$$\begin{cases} 2x+2y-z-2u-2v=1, \\ 3x+3y+z-5u-5v=1, \\ 4x+4y-2z-3u-3v=11? \end{cases}$$

Answer. 96; 140.

3. Bijection. Combinatorial Bijection Principle

Suppose that 59 teams are participating in a soccer cup. How many matches will be played?

Even after additional explanations regarding the rules of the tournament, a large number of respondents hesitated to provide the answer, attempting the construction of various schemes and the related calculations. There were mathematicians among those who got confused about this issue, not to mention those who participate in the competition schedule. This is a kind of question to which the student can give an instant and reasonable answer, and at the same time it can make the specialist lose his balance and dig deep in search of the truth that is right on the surface. A foreword regarding the rules of cup competitions is needed. The classic system is that each match should end effectively (that is, by the victory of one of the teams), and the team that loses is no longer taking part in the tournament. This is the fundamental rule of the winner's detection system, which is

called a single-elimination, knockout, or sudden death tournament. The rest of the rules are not significant. Therefore, they are the responsibility of organizers of the competition (for example, football association). The organizers compile a schedule of the tournament, providing the rules for the creation of pairs at different stages of the competition, decide on which stage one or another team enter the tournament etc. They can also make a decision that the teams should play two matches on each stage instead of one. This does not change the essence of the knockout system, provided that after these two matches one of the two teams necessarily leaves the tournament. This alternative rule does not change our task either: the answer is simply doubled.

Therefore, assume we have a “classic competition”, when two teams play one match to determine which one of them is eliminated. How many matches will have to be played by all the teams?

The one, who focuses from the beginning on the various options of the schedule of competition, will waste a good deal of time searching for the answer. And this is the most popular route to a solution. Alternatively, the one, who realizes that the schedule of the competition is irrelevant to the task, no matter how simple or tricky it is, will get the answer almost immediately. The only important rule is the following: the losing team is eliminated from the competition. Imagine that the tournament is over. Which teams *have not been knocked out*? Only the cup winner. All the rest were eventually defeated and *left the competition*. There are 58 of them. And there were the same amount of matches, because each team lost in a single match, and each match resulted in a defeat of one of the teams. The teams, which lost in the tournament, are in such connection with the matches played, that there is no doubt that the number of matches and the number of losing teams are the same. This connection is called bijective correspondence (or one-to-one correspondence). We will have to deal with many more similar situations and use the term ”bijective correspondence” or simply ”bijection”, and therefore, it is time to stop and explain in detail its exact meaning.

Let us have two collections (or, as mathematicians say, two sets) of some objects (they are called the elements of collections (sets)). The nature of these objects (elements) can be arbitrary. For example, it can be things, living beings, numbers, letters, geometric shapes, etc. What is essential about these objects, is that the elements of the same set should differ from each other. We denote our sets with the letters A and B. Imagine now that, we invented some rule and created pairs of elements (element from A; element from B), guided by this rule. Assume that each element of each set was used exactly once. In other words, we matched the elements of the sets A and B in such a way that each element of the set A received a partner from the set B and similarly, each element of the set B received a partner from the set A. This type of correspondence between the elements of the sets A and B is called a bijection between these two sets. In the above example, there is a bijection between the set of losing teams and the set of played matches. The rule of bijective correspondence is the following: each team corresponds to the match, in which it has lost.

Bearing in mind combinatorial applications, we will only deal with finite sets, that is, those containing a finite number of elements. For example, the set of letters in the English alphabet, the set of chess players ranked grandmaster, the set of three-digit natural numbers, etc. In the case of a finite set, the question about the number of its elements is relevant and well-posed. The answer is always some positive integer. On the contrary, infinite

sets contain an infinite number of elements. Such sets are created by humans' minds, and therefore, the examples can be found only in mathematics. The simplest among them are the sets of all natural numbers. In mathematics, we mainly deal with infinite sets. For instance, the sets of all real or all rational numbers, the set of all points of a line segment or all points of the plane, set of all lines of a plane or space, the set of all planes of space, the set of all planes parallel to a given plane, etc.

The notion of bijection introduced above applies to finite sets as well as to infinite. Applied to infinite sets, it produces a deep and elegant theory, which is of fundamental importance to modern mathematics. However, we will not give attention to this type of problem, moving straight to the outlined above aim – combinatorial applications.

If it is possible to establish a bijection between two finite sets, then these sets contain the same number of elements. Some scientists believe that this was the first mathematical principle that ancient people grasped in the times when they could not count, as there was no notion of numbers. The famous physicist Leon Cooper wrote: "The rocks piled up in the morning, with each rock being correspondent to one sheep, helped to determine if all the sheep have returned from a pasture in the evening. This simple and effective method is more primitive than the count. It only provides the ability to find out that there are as many rocks as there are sheep, disregarding the exact numbers." Letting sheep to the pasture one by one in the morning our ancestors piled up stones: one sheep – one stone. By doing so, they effectively established a bijection between the flock of sheep and the pile of rocks. In the evening, the procedure was repeated. If the bijection could not be established (there were extra stones) this time, then the owner quite reasonably believed that not all sheep returned home. If the flock and the pile were balanced, the owner knew that all the sheep returned. There is a bowl of peas and a bowl of beans. How to find if there are more peas than beans or otherwise? Of course, it is possible to count the grains and compare the resulting numbers. Alternatively, you can do what our ancestors did with sheep and stones. We take two more bowls and transfer peas into one of them, and beans into the second one, working with both hands simultaneously. As one of the initial bowls gets empty, we find the answer. In the case when both bowls get empty at the same time, we conclude that there is the same amount of peas and beans, although the exact number is unknown.

To find out how many visitors to expect on the late-night movie at the cinema, it is enough to check how many tickets have been bought. The information that all tickets are sold is a good reason for the director of the cinema to hope that all the seats will be occupied because the bijection between the tickets and the seats in the cinema hall has been established before the tickets have been sold. The above examples provide a sufficient illustration of the concept of bijection. In addition, note that counting is also based on the principle of bijective correspondence. Assume we count objects by saying "one", "two", "three", up to some number n. Acting in this way, we establish bijective correspondence between objects of a certain set with the initial n numbers of a natural series. The last (the highest) of the numbers used (the number n) is declared to be the number of objects. In essence, we assert that this set contains the same amount of objects as the interval of natural series from 1 to n inclusive.

As you can see, to count the objects of a set means to establish a bijection between this set and the reference set, created specifically for this purpose, namely, the part of the natural series.

What is the relationship connecting the concept of bijection of sets and combinatorial calculations? And what is the combinatorial bijection principle? If it is possible to establish a bijection between two finite sets, then these sets have the same quantitative composition. Widely discussed above, this property is called the combinatorial bijection principle. But can this self-evident statement prove useful in combinatorics? If so, in which situations? We have already had the opportunity to make sure that the answer to the first question is positive. Recall the example of 59 teams participating in a football knockout tournament. How many matches have to be played to identify the winner? The answer does not seem straightforward. Moreover, not having focused properly on the situation, one can not realize that the number of matches does not depend on the way the competition is scheduled. The number of possible schedules is incredibly large. Even if we choose a specific schedule of competition and count how many matches it presumes, this will not be the answer to the original question, because it applies to all possible tournament schemes. An exhaustive answer must either contain proof of the fact that the total number of matches does not depend on the schedule of the competition or be in the form of a table, specifying the dependence of the number of matches on the schedule of the tournament. As it has been shown above, the appreciation of the fact that there is a bijection between the played matches and the losing teams is the key to the answer. This guess, this little discovery solves the task because there is no doubt about the number of teams that have lost in the cup. It is 58. Therefore, we refer to the bijection to conclude that there have been 58 matches played.

The path from the original question to the final answer fits into the following scheme.

We need to count the elements of the set A. What hinders us is that the set is abstrusely arranged. Although we are able to separate its elements from each other strictly, we can not match them with natural numbers. And this is a great time to make the rescuing observation. We notice that there is a bijection between the original set and another set, which is arranged transparently enough (or we perceive it to be so) to count its elements. Such a bypass route allows us to find the number of elements of the set.

Primitive at first sight, this scheme is often very effective. The following examples serve to prove it.

Example 1.18. *A triangle is called integer-valued if all its sides are of integer lengths. How many integer-valued triangles are there with the sum of lengths of the shortest and the longest sides being equal to 12?*

We suggest a pair of solutions basing on the notion of bijection. The first one provides a simple geometric model of the problem, while the second has the advantage of being easily applied to the generalization of the original task, when 12 is replaced by n.

Solution I. As we usually do when compiling equations, denote the unknown sides of the triangle by letters. In our case, it is appropriate to denote the shortest side with x, and the middle one with y. We do not need to introduce additional letter for the longest side, as it is $12-x$ according to the statement of the problem. Note that the triangle may have sides of the same lengths (two or all). By calling aside “the shortest side”, we mean that there is no shorter side in the triangle. The same applies to the longest side. So using the introduced notation, we have:

$$0 < x \leq y \leq 12 - x. \tag{1.4}$$

The first inequality means that any side should have a positive length. Even if condition (1.18) is satisfied for the triplet of natural numbers $(x;\, y;\, 12-x)$, the existence of a triangle with such sides is not guaranteed. For example, it is impossible to create a triangle from the intervals of lengths 2, 4, and 10. Three intervals may compose a triangle if and only if the longest of them is shorter than the sum of two others (the triangle inequality). Hence, in addition to inequalities (1.18), there is another inequality, which has to be fulfilled:

$$x+y > 12-x.$$

Now, we list all the conditions, under which the triplet of natural numbers

$$(x;\, y;\, 12-x),$$

arranged in the increasing order, define an integer-valued triangle.

$$\begin{cases} x \text{ and } y \ - \text{ are natural numbers;} \\ y \geq x; \\ y \leq 12-x; \\ y > 12-2x. \end{cases} \tag{1.5}$$

At this point, we have the original problem reformulated. Really, what is the system of inequalities (1.5) about? It is about finding natural (having natural components) solutions to the system of inequalities

$$\begin{cases} y \geq x, \\ y \leq 12-x, \\ y > 12-2x. \end{cases} \tag{1.6}$$

Which is the relationship between the set of solutions to this system and the set of triangles from the statement of the problem? The answer is obvious: the correspondence

$$(x;\, y) \leftrightarrow (x;\, y;\, 12-x) \tag{1.7}$$

establishes a bijection between these two sets. The amount of suitable triangles is the same as the amount of solutions to the system of equations (1.6).

In a certain sense, we have "algebrated" the initial problem by changing the task, so that now we need to find the number of solutions of a certain algebraic system of inequalities with two variables. But have we get closer to the answer? Now, everything depends on how we can handle the system (1.6). The inequalities are linear and not complicated. There are different ways to deal with them. Exploring the options, one would come across the idea to turn to the geometric images of inequalities. This sound idea will quickly lead to the answer.

Every linear inequality with two variables correspondent to some half-plane of the coordinate plane. In particular, the inequality $y \geq x$ corresponds to the half-plane bounded from below with the line $y = x$. What does word "corresponds" mean here? It means that the inequality fulfills for all points of this plane and only for them. It does not fulfill for any other points. Similarly, the inequality $y \leq 12-x$ fulfills for all points of the half-plane bounded from above with the line $y = 12-x$ (and for points of this line too), and the inequality

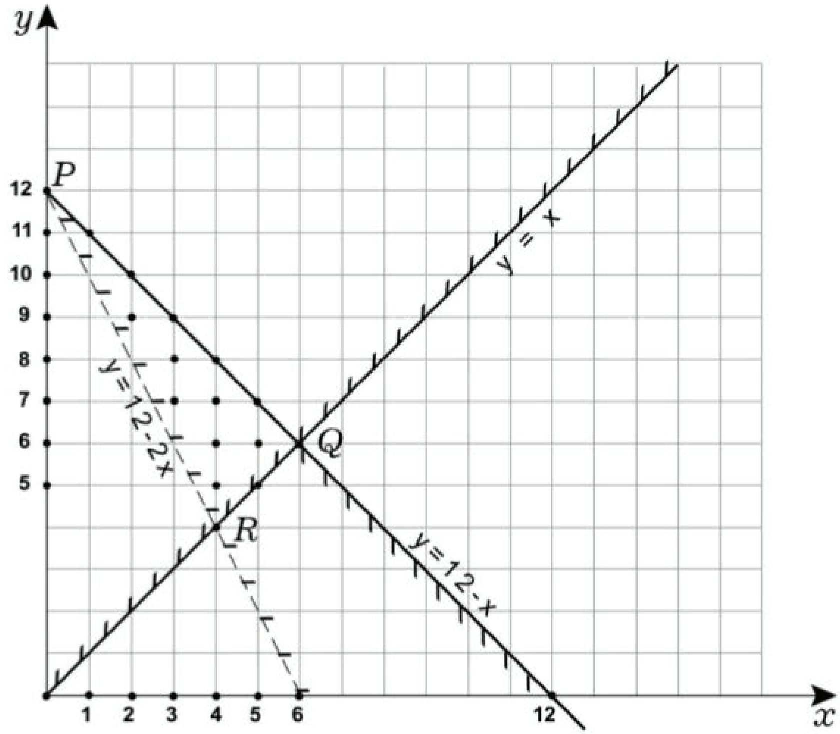

Figure 1.11. Geometric interpretation of linear inequalities.

$y > 12-2x$ fulfills for the points of the half-plane lying above the line $y = 12-2x$ (see Fig. 1.11). The intersection of these three half-planes, which is the triangle *PQR* including its sides *PQ* and *QR*, and excluding the points of the side *PR*. Every point inside this triangle (pair of its coordinates) is a solution to the system of equations (1.6), and any other point of the plane is not a solution. Recall that we are interested in natural solutions only (1.6), as they are in bijective correspondence with the required triangles. These solutions correspond to the points with natural coordinates inside the triangle *PQR*, and this correspondence is also bijective. Now, we have established the bijection between the required triangles and the integer points of the triangle *PQR*. It is straightforward to count such points – there are 14 of them. And this number is the answer to the problem.

Besides, we can provide the full list of the wanted triangles if required. According to the bijection (1.7), in order to do this, we need to list the coordinates of all integer points of the triangle *PQR*, adding to each of them the third number, which is the length of the longest side. Here is such a list:

$$\begin{array}{ccc} (1;11;11) & (2;9;10) & (2;10;10) \\ (3;7;9) & (3;8;9) & (3;9;9) \\ (4;5;8) & (4;6;8) & (4;7;8) \\ (4;8;8) & (5;5;7) & (5;6;7) \\ (5;7;7) & (6;6;6) & \end{array}$$

Solution II can be derived by alternative interpretation of inequalities (1.5). In the above arguments, we used the geometric interpretation of the inequality, eventually reducing the original problem to the counting of integer points of the coordinate plane inside the triangle PQR. For the second variant of solution, we confine ourselves to purely arithmetic sets. First of all, note that the length of the shortest side can vary from 1 to 6, that is, it can be any integer from the interval $[1, 6]$. It can not be greater than six, since summing it up with the length of the longest side we should get 12. Now, fix one of the possible values from 1 to 6. In order to prevent the use of additional letters, we denote this particular value by the letter x. Which are the possible options for the value of y (which is the length of the second shortest side of the triangle)? The answer to this question is given by the system of inequalities (1.6). We observe that it should exceed two values, x and $12-2x$, and thus, it should exceed the greater of them. We have to deal with two cases here. If $x > 12-2x$, that is, $x > 4$, then $y \geq x$. If $x \leq 12-2x$, that is, $x \leq 4$, then $y > 12-2x$. Taking now into account that the inequality $y \leq 12-x$ holds for any x, we get:

$$12-2x < y \leq 12-x, \text{ if } 1 \leq x \leq 4,$$

$$x \leq y \leq 12-x, \text{ if } 4 < x \leq 6.$$

There are $(12-x)-(12-2x) = x$ integer points in the interval $(12-2x, 12-x]$ and $(12-x)-(x-1) = 13-2x$ integer points in the interval $[x, 12-x]$. Hence, for $x \in [1, 4]$, there are x wanted pairs $(x; y)$, and for $x \in [5, 6]$, there are $13-2x$ wanted pairs. Therefore, there are

$$(1+2+3+4)+(3+1) = 14.$$

such pairs overall.

We have arrived at the same answer as before.

Which are the advantages of one method of solution over the other?

The most attractive feature of the first solution is its transparency and visualization. On the contrary, the second method lends itself to generalization easily. Once we change the number 12 *for the arbitrary* fixed number m, we get serious (though surmountable) complications in the application of the first method. And purely arithmetic approach does not suffer any essential changes, which we check below.

So we need to calculate the amount of different integer-valued triangles, where the sum of the shortest and the longest sides equals m. Let x and y be defined as before. In this case, we get the following inequalities:

$$\begin{cases} y \geq x, \\ y \leq m-x, \\ y > m-2x. \end{cases} \tag{1.8}$$

The natural solutions $(x; y)$ to this system are in bijection correspondence with the wanted triangles $(x; y; m-x)$. Hence, the question about the amount of triangles has transformed into the question of the amount of natural solutions to the system (1.8).

Fix any natural number x from the interval $\left[1, \left[\frac{m}{2}\right]\right]$ (the symbol $[a]$ denotes the integer part of the number a. In other words, this is the greatest integer which does not exceed

a). How many solutions to the system (1.8) are there, having the first component equal to x? The second component should belong to the intervals $[x, m-x]$ and $(m-2x, m-x]$ simultaneously. Thus, it should lie in their intersection (their common part). If $x > m-2x$, that is, $x > \left[\frac{m}{3}\right]$, then the intersection is $[x, m-x]$; if $m-2x \geq x$, then it is $(m-2x, m-x]$.

Consider these cases separately.

Let x belong to the interval $\left[1, \left[\frac{m}{3}\right]\right]$. Then y is a natural number from the interval $(m-2x, m-x]$, which yields that there are $(m-x)(m-2x) = x$ different values for y. Hence, there are

$$1+2+3+\ldots+\left[\frac{m}{3}\right] = \frac{1+\left[\frac{m}{3}\right]}{2} \cdot \left[\frac{m}{3}\right]$$

natural solutions to the system (1.8) with x belonging to the interval $\left[1, \left[\frac{m}{3}\right]\right]$.

Now, let x be in $\left(\left[\frac{m}{3}\right], \left[\frac{m}{2}\right]\right]$. Then y is a natural number from the interval $[x, m-2]$, and therefore, there are $(m-x)-(x-1) = m-2x+1$ possible values of y. Hence, there are

$$\left(m-2\cdot\left(\left[\tfrac{m}{3}\right]+1\right)+1\right)+\left(m-2\left(\left[\tfrac{m}{3}\right]+2\right)+1\right)+\ldots+\left(m-2\left[\tfrac{m}{2}\right]+1\right) =$$

$$= (m=1)\left(\left[\tfrac{m}{2}\right]-\left[\tfrac{m}{3}\right]\right)-2\cdot\frac{\left[\frac{m}{3}\right]+1+\left[\frac{m}{2}\right]}{2}\cdot\left(\left[\tfrac{m}{2}\right]-\left[\tfrac{m}{3}\right]\right) =$$

$$= \left(\left[\tfrac{m}{2}\right]-\left[\tfrac{m}{3}\right]\right)\cdot\left(m-\left[\tfrac{m}{2}\right]-\left[\tfrac{m}{3}\right]\right)$$

natural solutions to the system (1.8) with x belonging to the interval $\left(\left[\frac{m}{3}\right], \left[\frac{m}{2}\right]\right]$.

Finally, the system of inequalities (1.8) has

$$\frac{1}{2}\left(\left[\frac{m}{3}\right]+1\right)\cdot\left[\frac{m}{3}\right]+\left(\left[\frac{m}{2}\right]-\left[\frac{m}{3}\right]\right)\cdot\left(m-\left[\frac{m}{2}\right]-\left[\frac{m}{3}\right]\right)$$

natural solutions.

Particularly, if $m = 12$, then $\left[\frac{m}{3}\right] = 4$, $\left[\frac{m}{2}\right] = 6$, and the general formula yields the same result as the previous solutions which are 14.

Example 1.19. *Let n be an odd natural number, T_n be the set of all different integer-valued triangles with a perimeter of n, and T_{n+3} be the set of all different integer-valued triangles with a perimeter of $n+3$. Establish a bijection between the sets T_n and T_{n+3} to ensure that they contain the same amount of triangles.*

Solution. We remind that it is a good practice to try your best at solving problems by yourself before checking the suggested solution.

First, we note that if n is even, then the stated result does not hold. At least, there are even values of n for which the amounts of triangles in the given sets are different (actually, this holds true for every even n). For example, no integer-valued triangles with a perimeter of 2 exist, and there is one triangle with a perimeter of 5 (having sides (1; 2; 2)). Proceed with $n = 6$. There is only one integer-valued triangle with a perimeter of 6, which is (2; 2; 2). And 3 triangles are having a perimeter of 9: (3; 3; 3), (2; 3; 4), and (1; 4; 4).

The situation is different for the odd values of n. The table presents the composition of sets T_n and T_{n+3} for several odd values of n.

Table 1.10. Amounts of elements in T_n and T_{n+3}.

n	1	3	5	7	9	11
T_n	$\emptyset$	(1; 1; 1)	(1; 2; 2)	(1; 3; 3) (2; 2; 3)	(3; 3; 3) (2; 3; 4) (1; 4; 4)	(1; 5; 5) (2; 4; 5) (3; 3; 5) (3; 4; 4)
T_{n+1}	$\emptyset$	(2; 2; 2)	(2; 3; 3)	(2; 4; 4) (3; 3; 4)	(4; 4; 4) (3; 4; 5) (2; 5; 5)	(2; 6; 6) (3; 5; 6) (4; 4; 6) (4; 5; 5)

In addition to the illustration of the equality of amounts of elements in the sets T_n and T_{n+3} for small odd values of n, this table 1.10 contains a hint about the way, in which a bijection could be established between these sets in the case of arbitrary odd n. Adding 1 to every number in the row for T_n, one gets the numbers from the row for T_{n+3}.

This property is the basis for the following hypothesis:

If all sides of all the triangles from T_n (n is an odd number) are increased by 1, then the resulting triangles are all the triangles from T_{n+3}.

It seems very likely that this is the case. However, at the moment this is just a hypothesis, which requires thorough verification.

First of all, we need to investigate the following: is that true that if a, b and c are the sides of a triangle, then a triangle can be constructed out of the intervals $a+1$, $b+1$ and $c+1$? Recall a well-known fact that a triangle can be constructed out of three intervals if and only if the sum of lengths of any two of these intervals is less than the length of the third one. In our setting, this requirement is fulfilled. Really, the inequality

$$a+b>c$$

immediately yields that

$$(a+1)+(b+1)>c+1$$

(the other two cases are similar).

There is an encouraging, yet not final, result: if we increase all sides of all the triangles with a perimeter of n by 1, then we get the same amount of triangles with perimeters of $n+3$. Why the amount is the same? Because it is obvious that two different triangles both having perimeters of n will not transform into the same triangle with a perimeter of $n+3$, when their sides are increased by 1. We make the following conclusion.

The amount of triangles with the perimeter $n+3$ is greater than or equal to the amount of triangles with the perimeter n.

Besides, it can be seen from the proof that the result depends on n being odd, or even, or non-integer at all.

Now, we have approached the solution. Although the lasts steps are at the same time the toughest. From now on we assume that n is odd. Compare the sets of triangles T_n

(those having integer sides and a perimeter of n) and T_{n+3} (triangles with integer sides and a perimeter of $n+3$). We have learned already that by increasing all sides of all the triangles from T_n by 1, one gets the same amount of triangles from T_{n+3}. The next task is to make sure that there are no other triangles in T_{n+3}. How the appropriate check could be run technically? We need to take any triangle $(u; v; t)$ from T_{n+3} and prove that it can be constructed by lengthening the sides of some triangle from T_n. Clearly, the only candidate is the triangle $(u-1; v-1; t-1)$. But is this a triangle? This is the crucial question. If the answer is positive, then the problem is solved.

What do we know about u, v, t?

Firstly, u, v, t are natural numbers (or in other words, intervals, lengths of which are natural numbers).

Secondly, we may assume that $u \leq v \leq t$. This assumption is correct and reasonable, as it shortens the technical part of the proof.

Thirdly, $u+v > t$ (the triangle inequality, which justifies two other similar inequalities).

Lastly, $u+v+t = n+3$ (which is even number).

And what can we expect from the numbers $u-1$, $v-1$ and $t-1$? We expect that they have the *similar* properties with $n+3$ being replaced for n.

It is obvious that

1. $(u-1)+(v-1)+(t-1) = n$;

2. $u-1 \leq v-1 \leq t-1$.

There two more questions left:

a) Are the numbers $u-1$, $v-1$ and $t-1$ positive?

b) Does the inequality

$$(u-1)+(v-1) > t-1$$

hold?

Let us answer question a). If some of the numbers are not positive, then one of them should be $u-1$ (being the smallest of all three). But it can not be negative (because u is natural). Hence, $u-1 = 0$ and $u = 1$. Is it possible? It may seem it is. However, taking a closer look at the whole picture, one can conclude that there is no way for this to happen. Indeed, if the triangle has a side of 1, then the difference of two other sides is $t-v < 1$ (the triangle inequality). Taking into account that $t \geq v$ and the numbers t and v are natural, we get $t-v = 0$. Hence, the triangle $(1; v; t)$ is equilateral $(v = t)$. But the perimeter of such a triangle is $(2v+1)$, which is an odd number, and the perimeter of the triangle $(u; v; t)$ should be even. We arrive at a contradiction. Our assumption $u-1 \leq 0$ is its source. Therefore, $u-1 > 0$, and all the numbers $u-1$, $v-1$ and $t-1$ are natural.

It remains to prove the inequality

$$(u-1)+(v-1) > t-1.$$

Rearrange it to be

$$u+v-t > 1.$$

If this inequality does not hold, then

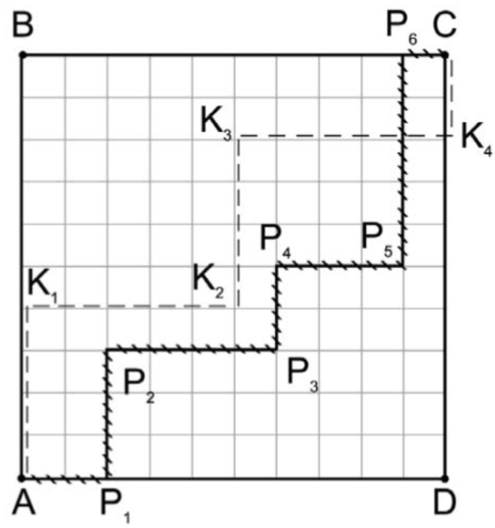

Figure 1.12. Paths on a squared paper.

$$u+v-t=1 \tag{1.9}$$

(as $u+v>t$ and all the numbers are natural). The last equality and the equality

$$u+v+t=n+3$$

yield that

$$t=\frac{n}{2}+1,$$

which is impossible, because t is integer, and n is odd.

The proved properties of the numbers $u-1$, $v-1$ and $t-1$ suggest that there is a triangle, which has sides of such lengths.

Hence, we completed the proof of the fact that the amount of integer-valued triangles with a perimeter of n (n is odd) and the amount of integer-valued triangles with a perimeter of $n+3$ are the same. In certain sense, the above proof is the best (or at least, the easiest one) among all other possible proofs. After all, the conclusion about the equality of amounts of triangles of different types is based on evident truth: the amounts are the same because we can come up with the rule ("plus 1 to each side"), which establishes a bijection.

Paths on a Squared Paper.

1. On a squared paper, there is a square with the side of 10 (which equals to lengths of 10 sides of small squares). Let A, B, C and D be the vertices of the square.

A spider can move along the lines that form the grid of the squared paper. He aims to get from the point A to the point C in the shortest possible way. What distance will the spider need to cover?

2. The spider has realized that the polygonal chains ABC and ADC are the shortest paths from A to C. However, there are many other shortest paths from A to C, e.g., $AK_1K_2K_3K_4C$

or $AP_1P_2P_3P_4P_5P_6C$. There are so many such routes with many of them interlacing that it is very difficult to deal with them. The spider eventually managed to enumerate the paths with the sequences of ten ones and ten zeros each: every path received its code and there left no more codes. In other words, he established a bijection between the shortest paths from A to C and the sequences of 20 symbols, ten of which are "1" and ten are "0". How did he do that? Which codes have the paths ABC, ADC, $AK_1K_2K_3K_4C$ and $AP_1P_2P_3P_4P_5P_6C$ received?

3. How many shortest paths from A to C are composed of three line segments?

4. How many shortest paths from A to C are composed of four-line segments?

5. What is the greatest amount of turns that the shortest path from A to C can have? How many shortest paths have the greatest amount of turns?

6. Establish a bijection between the shortest paths from A to C, which are composed of three line segments, and the paths composed of nineteen line segments.

Solution.

1. The spider always moves along horizontal (parallel to AD) or vertical (parallel to AB) lines. To get to the point C (starting from the point A), he needs to make 10 steps (units) to the right and the same amount of steps up (in other words, 10 steps to the east and ten steps to the north). Hence, the length of the shortest path is 20 units. To get to the point C using the shortest way he needs to adhere to the following rules:

a) from each intersection he has to move to the east or the north only (and never to the south or the west). This rule remains in force until he reaches the intervals BC or DC;

b) as soon as he reaches the interval BC he should move to the east only (to the point C);

c) as soon as he reaches the interval CD he should move to the east only (to the point C).

2. As we have already determined, reaching the point C from the point A via the shortest zigzag path means to make ten steps to the east and ten steps to the north. The exact path depends on the order, in which we take steps of these two types. Let the letter N denote that a step to the north is made and E denotes a step to the east. Then the sequence

$$NNNNNNNNNNEEEEEEEEEE$$

may be considered to be the code of the path which lies along with the intervals AB and BC. Every letter of the code means a step in a certain direction from one intersection to the other (the adjacent one). The codes of the paths $AK_1K_2K_3K_4C$ and $AP_1P_2P_3P_4P_5P_6C$ would be the following sequences of the letters N and E:

$$NNNNEEEEENNNNEEEEENN$$

and

$$EENNNEEEEENNEEENNNNE$$

respectively.

There is no doubt that every shortest path from A to C has a code attached to it, which consists of 10 letters N and 10 letters E. On the other hand, every such code corresponds to a certain path from A to C. There is a bijective correspondence between the shortest paths and the codes. And this correspondence is of essential value. Below, we will find

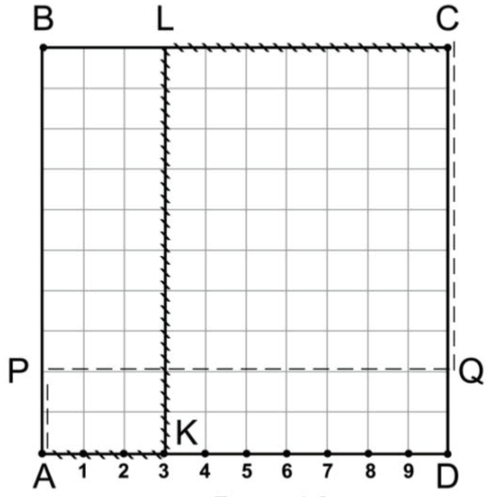

Figure 1.13. Paths with two turns.

the answer to the following question: how many shortest paths are there? The bijection established above will prove useful in this.

It remains to find out, where has the spider got the ones and zeros from. It is straightforward. If all the letters "*N*" are replaced with the digit 0, and all the letters "*E*" with the digit 1, then there appears a new system of codes, which differs from the previous one only visually. Conversely, one can change the letter "*N*" with the digit 1, and the letter "*E*" with the digit 0. Clearly, the letters "*N*" and "*E*" can be replaced by any two symbols. One just needs to establish the rule for translation of these symbols into moves on the squared paper.

3. Two such paths are drawn on Fig. 1.13 with dotted lines of two different types. There are two types of the paths composed of three line segments (or which is the same, the routes with two turns): those beginning with the horizontal segment (e.g., the path *AKLC*), and those beginning with the vertical segment (e.g., the path *APQC*). There are the same amounts of both because asymmetry with respect to the diagonal *AC* establishes a bijection between the paths of these two types. The path beginning with the horizontal segment is uniquely defined by the point on the side *AD*, in which it makes a turn into a vertical segment. This point in its turn is defined by a number from 1 to 9, by which inner integer points of the segment *AD* are denoted. This digit can be considered as a code of the corresponding path. For example, the code of the path *AKLC* is number 3. Hence, there are $2 \cdot 9 = 18$ wanted paths (paths with two turns).

4. As in the previous case, it is enough to count the paths beginning with the horizontal segment and then double up the result. In Fig. 1.14, two paths are beginning with the horizontal segment: *APQRC* and *AKLMC*. Such path can be defined by two numbers from the set $\{1, 2, 3, \ldots, 9\}$. The first of them defines the point on the interval *AD*, in which the path turns up (northbound), and the second one defines the height, at which turn to the right (eastbound) I located. Every such pair of numbers defines the path, and vice versa, every path (having three turns and horizontal initial segment) defines a pair of numbers. For instance, the path *APQRC* has the code (3; 4), and the path *AKLMC* has the code (5; 8).

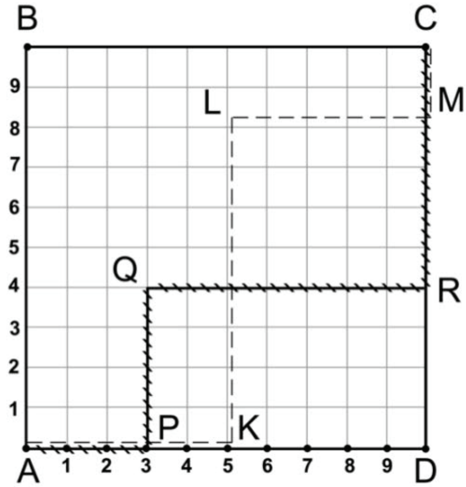

Figure 1.14. Paths with three turns.

There is the same amount of paths beginning with the horizontal segment, as there are codes (the combinatorial bijection principle), and there are $9 \cdot 9 = 81$ codes (the combinatorial rule of product). Hence, there are $2 \cdot 81 = 162$ paths with three turns.

Alternative Interpretation. Assume, we have defined a rectangle system of coordinates with the point of origin in the point A and the axes directed along the sides AD and AB. Then the code numbers of the path, say, $AKLMC$transform into the coordinates of the point L. Clearly, this is true for any path having horizontal initial segment. Every such path is uniquely defined by the second turning point (it is always inside the square). Hence, the amounts of integer points inside the square and paths having three turns and horizontal initial segment are the same. There are $9 \cdot 9 = 81$ integer points inside the square.

5. Every path is a polygonal chain, and the turns are its vertices. Every polygonal chain is a path from A to C. All polygonal chains are of the same length, and their segments are of integer length. It means that the amount of segments is maximal if all of them are the shortest, that is, of length 1 (if this is possible). Moving from the point A and turning after each step, one eventually reaches the point C, as after every two steps the path hits the diagonal AC. Hence, the polygonal chain with the greatest amount of segments has 20 of them. The amount of turning points is less by 1, that is, there are 19 of them. There are two such polygonal chains: one begins with the horizontal segment and the other begins with the vertical.

6. Arrange a bijection between the paths composed of three segments and the paths composed of nineteen segments, bounding ourselves to those paths, which begin with the horizontal interval. If a path is composed of three line segments, then the middle of them connects some inner point (not an ending point) of the interval AD with some point of the interval BC. Alternatively, if a path is composed of 19 intervals, then of them (necessarily vertical) has the length of 2, and others are of length 1. The desired bijection is established by matching two paths: one (composed of three segments) having the middle segment denoted by the number k ($k = 1, 2, ..., 9$), that is, having the code k (see step 3), and the

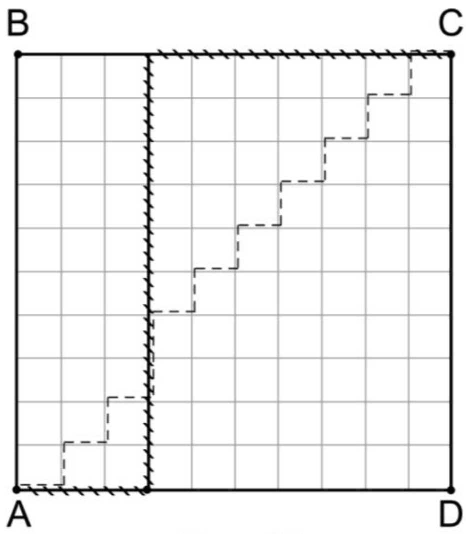

Figure 1.15. Bijection between paths.

second (composed of 19 intervals) having the segment of the length 2 lying on the segment of length 10 of the first path.

Fig. 1.15 represents two paths corresponding to each other by the above rule. It remains to note that the established bijection expands on the paths beginning with the vertical interval by the symmetry with respect to the diagonal AC.

Example 1.20.

1. Here we will deal with the summation of numbers and vectors. If one needs to calculate the sum of several (many) summands, then by the appropriate positioning of parentheses, this task can be reduced to the repeated summation of two summands. Moreover, the parentheses can be positioned in many different ways. The result does not depend on this. This is one of the fundamental arithmetical laws. It can be deduced from the associativity of addition, which refers to any three summands. The reader is well familiar with this property from the elementary school. Symbolically, it is presented as follows:

$$(a+b)+c = a+(b+c).$$

Considering the sums of many summands we will adhere to the following rule: each "+" sign must correspond to a certain pair of parentheses (opening and closing parentheses). Hence, there should be the same amount of pairs of parentheses as the amount of "+" signs in the expression. In particular, under such agreement, the associativity property is expressed as follows:

$$((a+b)+c) = (a+(b+c)).$$

Actually, we are not interested in associativity law and its consequences. We are dealing with a purely combinatorial problem: how many ways are there to place parentheses correctly in the sum of n summands? The word "correctly" here means that there should be

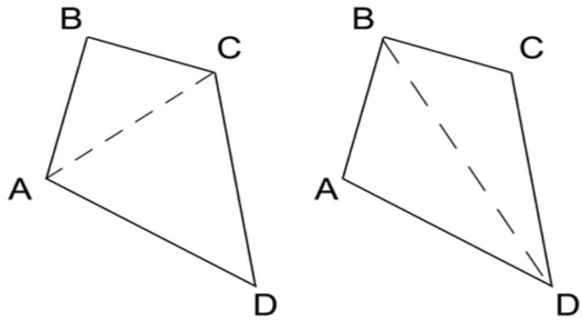

Figure 1.16. Split a quadrilateral into two triangles.

equal amounts of opening and closing parentheses, and every pair of parentheses (opening parenthesis; closing parenthesis) corresponds to a certain "+" sign. In other words, pairs of parentheses (opening and closing) must be in bijective correspondence with the "+" signs.

In the case of three summands, there are two ways to place parentheses:

$((a+b)+c)$ and $(a+(b+c))$.

Next, we provide below a full list of possible placings of parentheses in the sum of four summands:

$$\begin{array}{ll} (((a+b)+c)+d) & ((a+b)+(c+d)) \\ ((a+(b+c))+d) & (a+((b+c)+d) \\ (a+(b+(c+d))) & \end{array}$$

We highly recommend creating a similar list for the case of five summands. The resulting list should consist of 14 expressions. The readers who have followed this recommendation have ensured that even in the case of five summands, it is not straightforward to find all possible ways to place parentheses. Not to mention the cases of more summands. The question about the number of ways, in which parentheses could be placed in the sum of n summands appear to be quite complicated. We will be able to answer it later and will find out that there is an elegant formula for it. And now we are going to ascertain that this problem has a strong relationship with another no less interesting problems.

2. There are two ways to split a quadrilateral into two triangles with its diagonals (see Fig.1.16). A pentagon can be split by its diagonals into three triangles in five different ways (see Fig. 1.17).

There are fourteen ways to split a hexagon into triangles with its diagonals (see Fig. 1.18).

These findings are summarized in a table 1.11 (in the first row, there are numbers of sides of a polygon, and in the second we put the number of ways in which it can be split into triangles by its non-intersecting diagonals).

Surprisingly, the numbers in the second row are the same which we came across in the previous step. The corresponding table 1.12 follows.

Is this a sign of something more general than a random coincidence? If we had enough inspiration (and we should have when we get to work seriously), then sorting through all the options thoroughly we would make sure that the next numbers of the two tables are

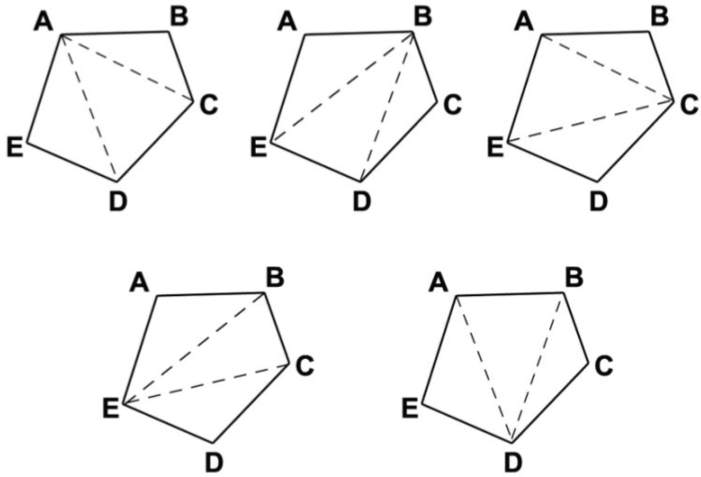

Figure 1.17. Split a pentagon into three triangles.

Table 1.11. Number of ways to split.

N of sides	4	5	6
N of ways to split	2	5	14

Table 1.12. Numberof ways to place parentheses.

N of summands	3	4	5
N of ways to place parentheses	2	5	14

also the same. Having six summands we can arrange parentheses in 42 ways. There is the same number of ways to split a heptagon into triangles with its non-intersecting diagonals (by the way, such a partition is called triangulation). It does not seem like a random coincidence. Comparison of the tables (expanded to include the latest result) suggests the way to formulate the hypothesis. It is better (more explicit) to express it with a formula having introduced the preliminary notation. Let $\alpha(n)$ denote the number of ways in which one can place parentheses in the sum containing n terms, and $t(n)$ denotes the number of ways in which one can perform the triangulation of an n-gon (a convex polygon having n sides). Our previous findings make the basis for the hypothesis:

$$\alpha(n) = t(n+1). \tag{1.10}$$

How can we prove it?

There are two approaches. We can attempt solving two problems independently: find the amount of ways to place parentheses in the sum of n summands (deduce the formula for $\alpha(n)$), and count the ways to make a triangulation of a convex n-gon (deduce the formula

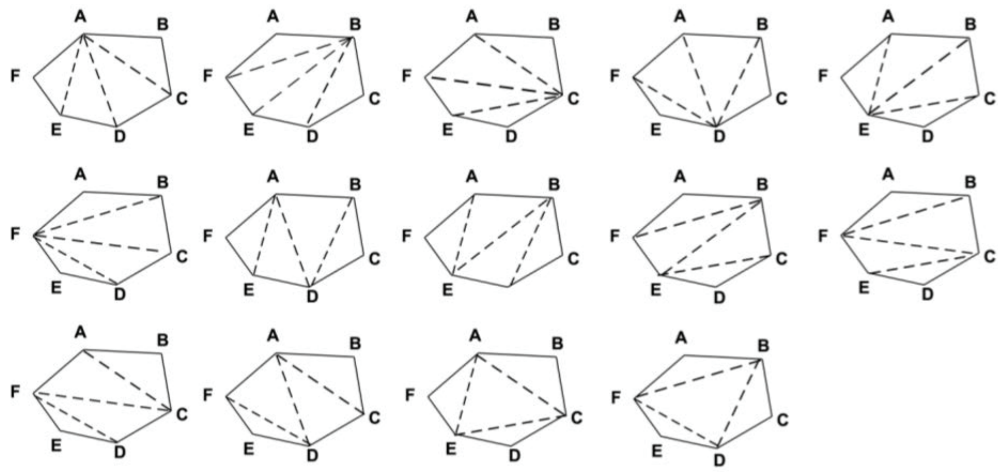

Figure 1.18. Split a hexagon into triangles.

for $t(n)$). Then we need to compare the resulting formulas for $\alpha(n)$ and $t(n+1)$.

The second way is much more interesting and expressive. If we were lucky enough to prove equality (1.10) without knowing the exact formulas for $\alpha(n)$ and $t(n)$, then we would avoid the necessity to deduce both of these formulas separately in the future. It would have been enough to find only one of them, and the second would have been received "for free" on the basis of equality (1.10). But how can one prove the equality of two quantities depending on the natural number n, if there is no idea how to calculate them, that is, there are no arithmetic formulas for them? Here is where the idea of bijection proves useful. Equality (1.10) will be proved if we are able to establish a bijective correspondence between the objects of two types: the algebraic sums of the symbols $a_1, a_2, ..., a_n$, which differ from each other in the placing of parentheses, and the various triangulations of a convex $(n+1)$ -angle. This is the type of situation, in which the combinatorial bijection principle is effective. When it is not clear how many objects of two types are there, but there is a hope that the amounts of objects of each type are equal, it is absolutely appropriate to attempt establishing a bijection between them. Success will confirm the assumption, though leaving the question about the number of items in each of the sets unanswered.

Now, we will prove the equality $\alpha(n) = t(n+1)$. Let us have a convex $(n+1)$-gon $A_1A_2A_3...A_nA_{n+1}$. Appropriately placing arrows on its sides A_1A_2, A_2A_3, ..., A_nA_{n+1}, we create the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_2A_3}$, ..., $\overrightarrow{A_nA_{n+1}}$. Their sum

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \overrightarrow{A_3A_4} + ... + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_{n+1}} \tag{1.11}$$

equals to the vector $\overrightarrow{A_1A_{n+1}}$. We arbitrarily (but correctly) place the parentheses in it. In doing so, we transform the operation of summation of n vectors into the chain of the length $n-1$ (according to the number of "+" signs) of additions of 2 vectors. According to the triangle law of vector addition, every of the $n-1$ additions geometrically means drawing of the diagonal in the polygon, and this diagonal does not intersect with the previously drawn diagonals. Overall there will be $n-1$ diagonals splitting the polygon into

the triangles. Hence, corresponding to every configuration of parentheses, there is a certain triangulation of the polygon. Conversely, there is a certain positioning of parentheses corresponding to every triangulation. Really, let we have some triangulation of the $n-1$-gon. Then at least one of the drawn diagonals along with two line segments of the polygonal chain $A_1A_2A_3...A_{n+1}$ bound a triangle. Let it be the line segments A_iA_{i+1} and $A_{i+1}A_{i+2}$. We place the first pair of parentheses in the sum (1.11) in such a way that it surrounds the sum of vectors $\overrightarrow{A_iA_{i+1}}+\overrightarrow{A_{i+1}A_{i+2}}$. After that the sum $\overrightarrow{A_iA_{i+1}}+\overrightarrow{A_{i+1}A_{i+2}}$ is replaced by the vector $\overrightarrow{A_iA_{i+2}}$, and the polygon $A_1A_2...A_iA_{i+1}A_{i+2}...A_{n+1}$ ($(n+1)$-gon) is replaced by $A_1A_2...A_iA_{i+2}...A_{n+1}$ (n-gon). The procedure is then repeated by the same rule for the second pair of parentheses and so on. Finally, we get some combination of parentheses, which in addition corresponds to the initial triangulation of the polygon. This yields that the established correspondence between the different configurations of parentheses in the sum of n summands and the different triangulations of a convex $(n+1)$-gon is bijective. This bijection proves that $\alpha(n)=t(n+1)$.

To make things more explicit, the bijective correspondence between the combinations of parentheses and the triangulations is presented on the scheme for the case of four summands and a pentagon.

3. Let us get back to the square split into 1×1 cells. This time we consider an *arbitrary* $n\times n$ square $ABCD$ (n is natural number) instead of specific 10×10 square. Again, we are interested in the shortest zigzag paths from A to C. However, here we limit ourselves to those paths, which do not cross the diagonal AC (though touching it is acceptable) and lie below it. How many different paths of this type are there? On Fig. 1.20, Fig. 1.21 and Fig. 1.22 all the paths are presented for the cases $n=2$, 3, and 4 respectively.

Denoting $p(n)$ the amount of "subdiagonal" paths from A to C in the square $n\times n$, construct a table 1.13 for the small values of n.

Table 1.13. Number of subdiagonal paths.

n	2	3	4
$p(n)$	2	5	14

We have three numbers 2, 5, and 14, which are the same as we have found in the previous two steps. In the previous steps, these numbers acted as the possible values of $\alpha(n)$ and $t(n)$. For it to be easier to compare the values of the variables (functions) $\alpha(n)$, $t(n)$ and $p(n)$, we construct the following table 1.14.

Table 1.14. Values of functions $p(n),\alpha(n),t(n)$.

n	3	4	5
$\alpha(n)$	2	5	14
$t(n+1)$	2	5	14
$p(n-1)$	2	5	14

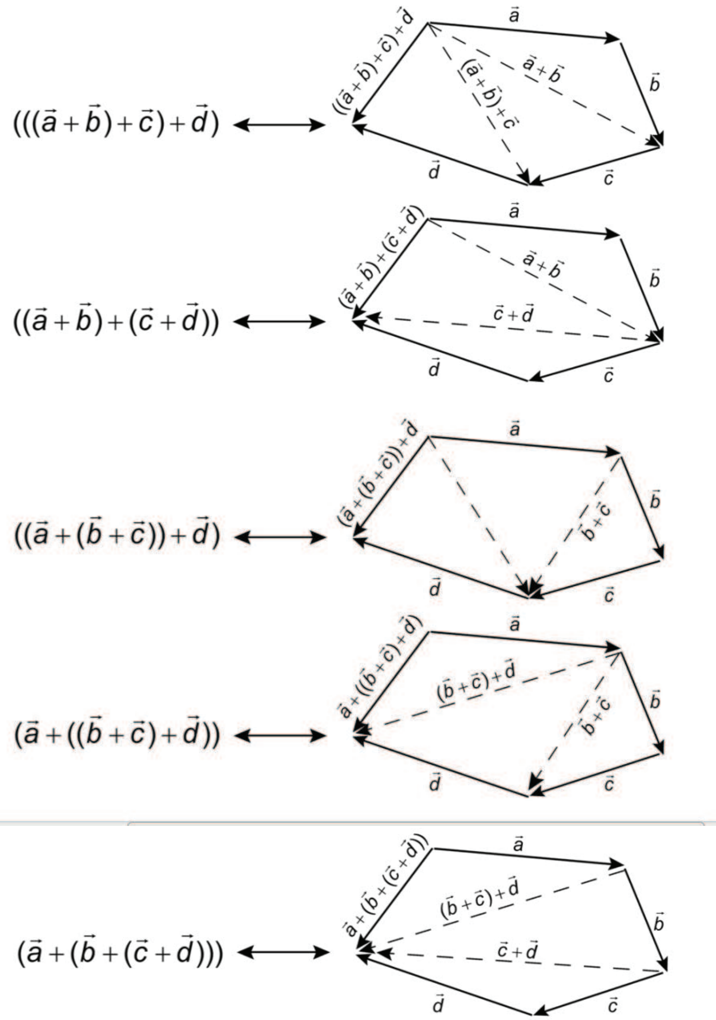

Figure 1.19. Correspondence between combinations of parentheses and triangulations.

It is clear from the table that the functions $p(n)$, $\alpha(n)$ and $t(n)$ gain the same values with an offset (a positive or negative lag) by one or two steps. We have already determined that the functions $\alpha(n)$ and $t(n+1)$ are equal, that is, they gain the same values for all natural values of n, not only for $n=3,4$ or 5. We reached this result by establishing a bijection between the different configurations of parentheses in the sum of n summands and the different triangulations of a convex $(n+1)$-gon. We can not be sure though if the equality

$$\alpha(n) = p(n-1).$$

holds for all n.

The table composed for three values of n $(n=3,4,5)$ is a basis for the assumption that for all natural values of n this weird equality holds. Really, why is the amount of the "subdiagonal" paths from A to C in the $(n-1)\times(n-1)$ square$ABCD$ equal to the number

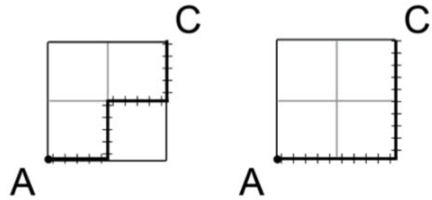

Figure 1.20. Shortest zigzag paths in 2×2 square.

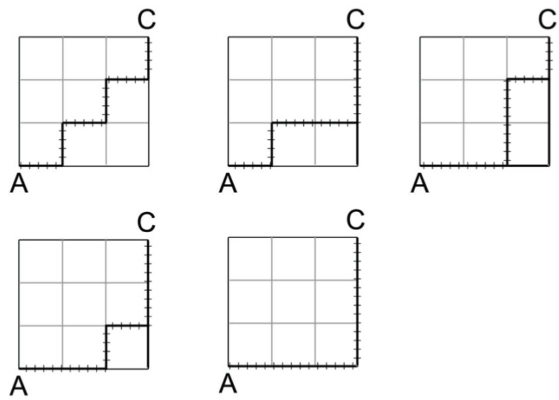

Figure 1.21. Shortest zigzag paths in 3×3 square.

of all possible correct combinations of parentheses in the sum of n summands? If this is true for any $n \geq 3$ (besides, it is correct for $n = 2$: $\alpha(2) = p(1) = 1$), then it is tempting to find the reasoning behind this equality.

Considering arbitrary shortest paths from A to C in a 10×10 square, we have found out that they can be encoded by the sequences of 20 letters, ten of which are N and ten are E. The letter N denotes a northbound move, while the letter E denotes an eastbound. Starting from the point A and making 10 steps to the north and 10 to the east we will finish in the point C. In doing so, we can randomly mix northbound and eastbound steps. Different combinations of northbound and eastbound steps result in different paths from A to C. There is a bijection between the shortest paths from A to C and the sequences composed of ten letters N and ten letters E, which we call the codes of the corresponding paths.

Obviously, in the same manner, the paths inside a $n \times n$ square can be encoded. The code of each such path comprises n letters N and n letters E.

In the current setting, we are interested only in those paths from A to C in the $n \times n$ square, which lie below the diagonal AC, that is, in the triangle ACD. Which characteristic property do the codes of such paths have? (Saying "characteristic property" we mean such property of the codes of the "subdiagonal" paths, which distinguishes them from the codes of all other paths). This is a simple question. Assume, we start from the vertex A and move along the shortest path to the vertex C step by step . What is the condition under which we will not get over the diagonal AC during the whole journey? The points of the triangle ACD

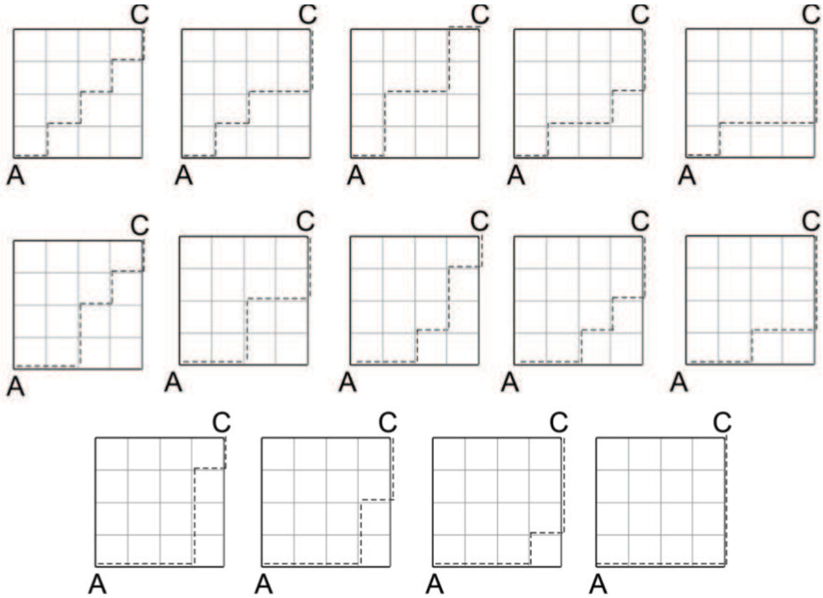

Figure 1.22. Shortest zigzag paths in 4×4 square.

have the following property: the distance to the interval AD from each of them does not exceed the length of the interval AB. This property is characteristic, because the distance from any point of the triangle ABC to the interval AD is no less than the distance to the interval AB Fig. 1.23.

Hence, in order to stay in the triangle ACD on each stage of the path from A to C, the number of eastbound steps taken up to this stage should not be less than the number of northbound steps. (In the case the amounts of steps are equal, we are on the diagonal AC). This property of the "subdiagonal" paths is easily translated into the coding language. For our simplicity, we introduce the "segment of code" notion. We define it in the following way: a segment of code is a sequence of several letters standing in a row and beginning with the first letter of a code. For instance, the code $NEENEN$ has 6 segments: N, NE, NEE, $NEEN$, $NEENE$ and $NEENEN$. The last of them is the code itself. A path is "subdiagonal" if and only if every segment of its code contains no more letters N than the letters E. For example, the path with the code $EENENN$ is subdiagonal, while the one with the code $EENNNE$ is not. There is another way to define the code of subdiagonal paths: this is such code, in which the letter N never outruns the letter C. Alternatively: these are the codes having the letter E as a leader on each stage.

And now we will show that a similar coding system can be used to encode the combination of parentheses in the sum of n summands. It is appropriate to begin with an example. Let us have the sum of five summands with the following placing of parentheses:

$$(((a+b)+(c+d))+e). \tag{1.12}$$

Imagine that all the closing parentheses and summands have been removed. What does remain from the above expression? There is a sequence composed of four opening paren-

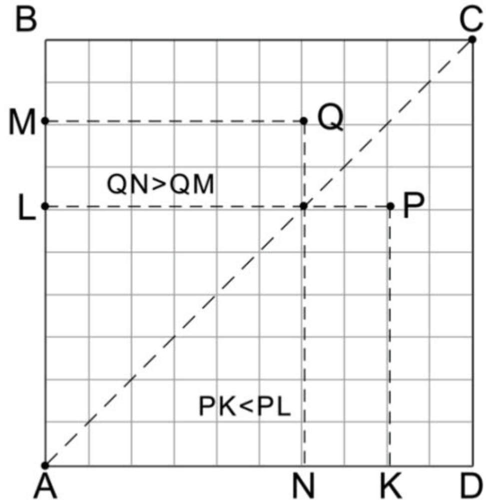

Figure 1.23. "Subdiagonal" paths.

theses and four "+" signs. Here is this sequence:

$$(((+ + (+ + \tag{1.13}$$

Can this sequence act as a code for the initial combination of parentheses? This is the main question for us. However, first, we need to answer an auxiliary, yet very important question: which sequences of opening parentheses and "+" signs (in equal quantities) can act as such codes? Really, the sequence

$$+ + + + (((($$

can not correspond to any of the combinations of parentheses in the sum of five summands. Actually, any such combination must begin with an opening parenthesis, so its code (created by the removal of right parentheses and summands) can not start with "+" sign.

Every pair of parentheses (opening and closing parentheses) in a correct combination of parentheses is connected to some "+" sign. And this "+" sign must necessarily be placed after the opening parentheses. This means that if all closing parentheses and summands are removed, then the resulting sequence of opening parentheses and "+" signs the latter never outrun the former. In other words, in every segment of such sequence, derived as above, the number of opening parentheses is greater than or equals to the number of "+" signs. Now we need to check if every sequence created by k opening parentheses and k "+" signs, where the former outruns the latter, can be derived from some combination of parentheses in the sum of $k+1$ summands. Choose any such sequence. We will ensure that it uniquely defines the correct combination of parentheses in the sum of $k+1$ summands. Note that the combination of parentheses does not depend on the way the summands are denoted. Therefore, recovering the combination from its code (hereinafter we call the sequences of opening parentheses and "+" signs beginning with the opening parenthesis, the code) we will use random letters.

Let us have an arbitrary sequence of symbols "(" and "+" (k of each), which begins with "(". Let us find the last parenthesis. It should be followed by at least one "+" sign, because the parenthesis leads the sequence. So the sequence is of the form: $\ldots(+\ldots+$.

The "partner" of the last parenthesis is the "plus", which immediately follows it, so the combination of parentheses in this part is given by the expression:

$\cdots(a+b)+\cdots+$ (there may be no "plus" after the closing parenthesis). Now, remove the expression $(a+b)$ from this sequence to get the sequence consisting of $k-1$ parentheses and $k-1$ "+" signs. The parenthesis is also a leader here, hence the procedure of recovering the whole combination can be continued, considering $(a+b)$ as an elementary summand positioned between the "+" signs adjacent to it or at the end of the expression (after the last "+" sign). At each step, we neutralize the opening parenthesis by the closing one, so after k steps we arrive at the complete combination of parentheses.

Example 1.21.

Let us have the code

$(((+(++(++$.

The positions of summands are recovered unambiguously (between the pairs "(+" and "++" and at the end of the code):

$$(((a+b+(b+c+(d+e+f.$$

Further, using the introduced rule

$$\begin{array}{ll}
(((a+(b+c+(d+e)+f & (d+e)=p \\
(((a+(b+c+p+f & \\
(((a+(b+c)+p+f & (b+c)=q \\
(((a+q+p+f & \\
(((a+q)+p+f & (a+q)=t \\
((t+p+f & \\
((t+p)+f & (t+p)=s \\
(s+f & \\
(s+f) & \\
\text{------------} & \\
((t+p+f) & \\
(((a+q)+p)+f) & \\
(((a+(b+c))+p)+f) & \\
(((a+(b+c))+(d+e))+f). &
\end{array}$$

Let us summarize our reasoning. There is a bijection established between the set of combinations of parentheses in the sum of $k+1$ summands and the set of sequences, composed of k symbols "(" and k symbols "+", beginning with the symbol "(".

If we replace all the parentheses by the letter E and all the "pluses" by the letter N in the code of combination of parentheses composed of k symbols "(" and k symbols "+", then we get the code of a subdiagonal path in the $k\times k$ square.

That is why $d(k+1)=p(k)$, or equivalently,

$$d(n)=p(n-1).$$

Basing on the concept of bijection we were able to establish interesting relations between three problems, which at the first sight appear to be uncorrelated: the triangulation of a convex polygon, the combinations of parentheses, and the subdiagonal paths in a square. Actually, it is reasonable to add one more problem to this family: the sequences composed of k symbols, say, a and k symbols b, every segment of which contains no fewer symbols a than the symbols b. Let $l(k)$ denote the number of such sequences. We have proved the following equalities:

$$d(n) = t(n+1) = p(n-1) = l(n-1).$$

There is still no computational formula for any of these quantities (all of them are functions of the natural argument n). If we can deduce such a formula for any one of these functions in the future, then it will be appropriate for all other as well. In a short, while we will count the subdiagonal paths in the $n \times n$ square and will thus define the computational formula for $p(n)$.

Problems

Problem 1.99. *Establish a bijection between two-digit even numbers and two-digit odd numbers.*

Answer. For example, one can match every two-digit even number with the number exceeding it by 1.

Problem 1.100. *Is it possible to establish a bijection between all two-digit natural numbers and those three-digit numbers, the second digit of which is 1? What is the answer to the previous question if we consider three-digit numbers having 1 as the first digit instead of the second? What is the answer to the original if we consider three-digit numbers having 1 as the third number instead of the second?*

Answer. Yes; no; yes.

Problem 1.101. *How many ways are there to establish bijection between the sets A and B, each consisting of:*

a) two elements?

b) three elements?

c) four elements?

Answer. a) 2; b) 6; c) 24.

Problem 1.102. *Let us have an equation with two unknowns, x, and y. Its solution is the ordered pair of numbers* $(a; b)$*, which transform the equation into correct numeric equality when x and y are replaced by a and b respectively. The numbers a and b are called the first and the second components of the solution respectively. For example, solutions to the equation*

$$2x + 3y = 24 \tag{1.14}$$

are the pairs $(3;6)$, $(0;8)$, $(-9;14)$, $\left(-\frac{1}{2}; \frac{23}{3}\right)$ *and infinite amount of others. If both components of a solution are natural, then a solution is called natural. For instance, the natural solutions to equation* (1.14) *are* $(3;6)$, $(6;4)$, $(9;2)$ *(are there any other natural solutions?).*

We call a solution integer non-negative if both its components are integer non-negative numbers. For example, $(3;6)$ *and* $(0;8)$ *are integer non-negative solutions to equation* (1.14) *(find all its integer non-negative solutions).*
Prove that the correspondence

$$(a;b) \leftrightarrow (a+1;b+1)$$

Establishes a bijection between all integer non-negative solutions of the equation

$$x+y=n$$

and all natural solutions to the equation

$$x+y=n+2$$

(n is natural).

Problem 1.103. *1. Establish a bijection between two-digit numbers having the sum of digits equal to k* $(k=1,2,...,9)$ *and two-digit numbers having the sum of digits equal to* $19-k$.

2. For every natural k, find the amount of two-digit numbers, the sum of digits of which is k.

Answer. 1) We correspond the number $\overline{ab}$ *having the sum of digits equal to k* $(a+b=k)$ *to the number xy, where* $x=10-a$, $y=9-b$. *2) If* $x=1,2,...,8,9$, *then there are k two-digit numbers with the sum of digits equal to k. If* $k=10,11,...,17,18$, *then there are* $19-k$ *two-digit numbers with the sum of digits equal to k.*

Problem 1.104. *Let A be the set of those natural solutions to the equation*

$$x+y=n,$$

the second component of which is greater than or equal to the first, and let B be the set of those natural solutions to the equation

$$x+y=n+1,$$

the second component of which is greater than the first (n is a given natural number). Establish a bijection between the sets A and B.

Hint. If $(p;q)$ *is a solution to the equation* $x+y=n$ *and* $p \le q$, *then* $(p;q+1)$ *is a solution to the equation* $x+y=n+1$ *and* $p<q+1$. *Vice versa, if* $(t;s)\in B$, *then* $(t;s-1)\in A$.

Problem 1.105. *A solution to the equation with three variables* $x;y;z$ *is the (ordered) triplet of numbers* $(a;b;c)$, *which solves the equation. The latter means that replacing* x,y *and* z *by* a,b *and* c *respectively, transforms the equation into correct numeric equality. The numbers* a,b, *and* c *are called the first, the second, and the third components of the solution respectively.*

Let A be the set of those natural solutions to the equation

$$x+y+z=n$$

(n is given natural number), the components of which form an increasing sequence and B be the set of those natural solutions to the equation

$$x+y+z=n+3,$$

the components of which form a strictly increasing sequence.

Establish a bijection between the sets A and B.

Hint. For example,

$$(a;b;c) \leftrightarrow (a;b+1;c+2),$$

where $(a;b;c) \in A$.

Problem 1.106. *Generalize the above result for the case of equations with respect to k unknowns. If the first equation is*

$$x_1+x_2+x_3+\ldots+x_k=n,$$

what the second equation is?

Answer. The right-hand side of the second equation is

$$n+\frac{k(k-1)}{2}.$$

Problem 1.107. *Establish a bijection between integer non-negative solutions of the equation*

$$x_1+x_2+x_3+\ldots+x_k=n$$

(n is natural) and natural solutions to the equation

$$x_1+x_2+x_3+\ldots+x_k=n+k.$$

Hint. Refer to Problem 1.104.

Problem 1.108. *Let c be a given number from the interval $[0,9]$. Prove that the equation*

$$x+y+z=c+10$$

has three times more integer non-negative solutions, one of the components of which is greater than 9, than the equation

$$x+y+z=c$$

has integer non-negative solutions.

Solution. The problem is about the correspondence between the set of integer non-negative solutions to the equation

$$x+y+z=c \tag{1.15}$$

and the set of those integer non-negative solutions to the equation

$$x+y+z=c+10, \tag{1.16}$$

one of the components of which is greater than or equal to 10.

First, note that there are no solutions to the second equation that has two or three components greater than 9, because $c \leq 9$.

Now, we prove that those solutions to equation (1.16), the first component of which is greater than 9, are in "natural" correspondence with all integer non-negative solutions to equation (1.15). Really, let $(p; q; r)$ be a solution to equation (1.16). Then $(p+10; q; r)$ is a solution to equation (1.16), because from $p+q+r=c$ it follows that $(p+10)+q=r=c+10$. Conversely, if $(u; v; t)$ is a (integer non-negative) solution to equation (1.16) and $u \geq 10$, then $(u-10; v; t)$ is an integer non-negative solution to equation (1.15). Therefore, the correspondence

$$(p; q; r) \leftrightarrow (p+10; q; r)$$

There is a bijection between the two types of solutions.

Similarly, a bijection can be established between integer non-negative solutions to the equation (1.15)and those integer non-negative solutions to the equation (1.16), the second component of which exceeds 9, and also those integer non-negative solutions to the equation (1.16), the third component of which is greater 9.

Problem 1.109. *Local bus tickets from the same series are enumerated by three digits from 000 to 999.*

1. *How many bus tickets are there from the same series?*

2. *Let k be an integer from* 0 *to* 27*. Establish a bijection between the numbers of those tickets, the sum of digits of which is k, and another tickets, having the sum of digits of their numbers equal to* $27-k$.

Problem 1.110. *Let k be a natural number from the interval* $[1, 28]$*. Establish a bijection between three-digit numbers, the sum of digits of which is k, and three-digit numbers, having the sum of digits of their numbers equal to* $27-k$.

Hint. This problem differs from the previous one in that here we deal with three-digit numbers with the first digit being different from zero and in the previous case the question is about the arbitrary groups of three digits.

Solution. Let $\overline{\alpha\beta\gamma}$ be a three-digit number (α, β, γ is its digits, and hence, $\alpha \geq 1$). We match it with the number $\overline{xyz}$, the digits of which relate to the digits α, β and γ in the following way: $x = 10-\alpha$, $y = 9-\beta$, $z = 9-\gamma$. First, note that the numbers x, y and z are one-digit and $x \geq 1$ (as $\alpha \leq 9$), therefore for any number $\overline{\alpha\beta\gamma}$, there exists a three-digit number $\overline{xyz}$ corresponding to it. By our rule, the latter number will have the number $\overline{\alpha\beta\gamma}$ corresponding to it, as $\alpha = 10-x$, $\beta = 9-y$, $\gamma = 9-z$. Hence, our rule of correspondence combines all three-digit numbers into pairs. Those numbers forming a pair, correspond to each other. In particular, this proves that the established correspondence is bijective. Now, calculate the sum of digits of the number $\overline{xyz}$, assuming that the sum of digits if the number $\overline{\alpha\beta\gamma}$ is k $\quad(\alpha+\beta+\gamma = k)$. We get: $x+y+z = (10-\alpha)+(9-\beta)+(9-\gamma) = 28-(\alpha+\beta+\gamma) = \quad = 28-k$. This equality means that if one number of the pair has the sum of its digits equal to k, then for the other number of the pair, this sum equals to $28-k$. Hence, there are the same amounts of these numbers.

Problem 1.111. *How many three-digit numbers are there, where the third digit equals the sum of the first two?* *Answer.* *45.*

Solution. First Approach. One can erroneously suppose that the first two digits can be arbitrary and the third depends on them. This is correct if an additional condition is introduced: the first two digits should be small enough for their sum not to exceed 9. Two-digit numbers having the sum of their digits equal to $k \quad (1 \le k \le 9)$, are in bijective correspondence with those solutions of the equation

$$x + y = k,$$

which have a non-zero first component. There are k such solutions. This means that there are $1+2+3+4 = 5+6+7+8+9 = 45$ three-digit numbers with the required property. Second Approach. The sum of the first two digits should be less than 10. Hence, the amount of wanted numbers is the same as the amount of two-digit numbers having the sum of their digits less than 9. According to the result of Problem 3, there are half as many of these numbers as there are two-digit numbers altogether. Hence, there are 45 of them.

Problem 1.112. *Establish a bijection between natural solutions to the inequality*

$$x_1 + x_2 + x_3 + \ldots + x_k < n$$

($x_1, x_2, \ldots, x_n$ are unknown, n is a given natural number) and natural solutions to the equality

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} = n.$$

Solution. If $(\gamma_1; \gamma_2; \ldots; \gamma_k; \gamma_{k+1})$ is a solution to the equation, then $(\gamma_1; \gamma_2; \ldots; \gamma_k)$ is a solution to the inequality. Why? Because $\gamma_{k+1} > 0$. From the equality

$$\gamma_1 + \gamma_2 + \ldots + \gamma_k + \gamma_{k+1} = n,$$

it follows that

$$\gamma_1 + \gamma_2 + \ldots + \gamma_k = n - \gamma_{k+1} < n.$$

Conversely, if $(\beta_1; \beta_2; \ldots; \beta_k)$ is a solution to the inequality, that is

$$\beta_1 + \beta_2 + \ldots + \beta_k < n,$$

then the number $\beta_{k+1} = n - (\beta_1 + \beta_2 + \ldots + \beta_k)$ is positive, and thus, $(\beta_1; \beta_2; \ldots; \beta_k; \beta_{k+1})$ is a natural solution to the equation.

Therefore, if we "shorten" a solution to the equation by removing its last component, then we get a solution to the inequality; conversely, every solution to the inequality can be extended by one natural component to become a solution to the equation.

This establishes a bijection between the solutions of two types, which yields that there are the same amounts of them.

Problem 1.113. *Prove that the inequality*

$$x_1 + x_2 + x_3 + \ldots x_k < n$$

($x_1, x_2, \ldots, x_k$ are unknown; n is given natural number) has the same number of integer non-negative solutions as the equation

$$x_1 + x_2 + \ldots + x_k + x_{k+1} = n - 1.$$

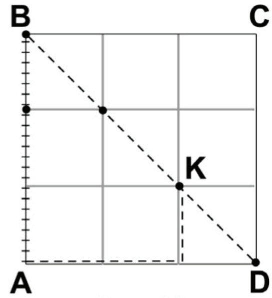

Figure 1.24. Shortest zigzag paths from the vertex A to points of the diagonal.

Problem 1.114. *Let A be the set of all those integer non-negative solutions to the inequality*

$$x_1 + x_2 + \ldots x_k < n,$$

the first component of which is non-zero, and B be the set of all integer non-negative solutions to the inequality

$$x_1 + x_2 + \ldots x_k < n - 1.$$

Establish a bijection between the sets A and B.

Hint. We match the solution $(1+\gamma_1; \gamma_2; \gamma_3; \ldots; \gamma_k)$ of the first inequality to the solution $(\gamma_1; \gamma_2; \gamma_3; \ldots; \gamma_k)$ of the second one. This correspondence is bijective.

Problem 1.115. *Let C be the set of those natural solutions to the equation*

$$x + y + z = t = n,$$

the first component of which is 1, and D be the set of those natural solutions to this equation, the third component of which is 1. Establish a bijection between the sets C and D.

Problem 1.116. *In the* 3×3 *square ABCD divided into cells with the side of 1, consider the shortest zigzag paths from the vertex A to the points of the diagonal BD (there are two such paths drawn on Fig. 1.24 ending in the points B and K respectively). Establish a bijection, which matches these paths with the vertices of some cube.*

Hint. The paths can be encoded by sequences of the length 3 composed of two symbols. A cube with edges of 1 can be placed in the coordinate space in such a way that one of his vertices is in the point of origin and three vertices adjacent to it lie in the points $(0;0;0)$, $(0;1;0)$ and $(0;0;1)$.

Problem 1.117. *How many different integer-valued (having integer sides, that is sides of integer lengths) triangles are there with the middle side of* 11*? "The middle side" is the side for which there is another side greater than or equal to it and one more side, which is shorter or equal to it. The examples of such triangles are* $(10;11;12)$, $(11;11;12)$, $(11;11;11)$, $(2;11;11)$, *etc. (in this notation, there are lengths of the sides of a triangle inside parentheses).*

Denote the shortest side of a triangle by $x \quad (x \in 11)$, *and its longest side by* $y \quad (y \geq 11)$.

Establish a bijection between all triangles of the form $(x; 11; y)$ *and some set of integer (having integer coordinates) points* $(x; y)$ *on the coordinate plane. What is this set?*

<u>*Answer.*</u> *The set of points of the coordinate plane, which are a bijective pairs for the wanted integer-valued triangles, is the set of integer points lying inside and on the sides of the triangle bounded by the lines* $x = 11$, $y = 11$ *and* $y = x + 10$. *There are 66 such points.*

<u>*Hint.*</u> *Construct the inequality of triangle, which guarantees the existence of a triangle with the sides* x, 11 *and* y, *where* $x \leq 11 \leq y$.

Problem 1.118. *Generalize the previous problem, assuming now that the length of the middle side of a triangle is n. How many integer-valued triangles are there this time?*

<u>*Answer.*</u> $\frac{1}{2}n(n+1)$.

Problem 1.119. *How many different integer-valued triangles are there with the sum of two longest sides being equal to* 24*?*

<u>*Answer.*</u> *30.*

<u>Hint.</u> Denote the shortest side by y and the middle one by x. Then the longest side is $24 - x$. In order for a triangle with the sides y, x and $24 - x$ (where $y \leq x \leq 24 - x$) to exist, the inequality of triangle

$$y + x > 24 - x$$

has to hold. Additionally, this condition is sufficient for the triangle to exist. Thus, we have a bijection between the wanted integer-valued triangles and all those pairs of natural numbers $(x; y)$, for which the following inequalities hold:

$$\begin{cases} y \leq x, \\ x \leq 24 - x, \\ y + x > 24 - x. \end{cases}$$

Transform it into equivalent system of inequalities:

$$\begin{cases} 24 - 2x < y \leq x, \\ x \leq 12. \end{cases}$$

Taking into account that we need only natural solutions, we can replace the inequality

$$24 - 2x < y$$

with

$$25 - 2x \leq y.$$

Therefore, we arrive at the following system of inequalities:

$$\begin{cases} 25 - 2x \leq y \leq x, \\ x \leq 12. \end{cases}$$

Its natural solutions $(x; y)$ are in bijective correspondence with the required integer-valued triangles

$$(y; x; 24 - x).$$

There are multiple ways to deal with this system of inequalities: we can move on to the geometric interpretation (on the coordinate plane) or calculate the number of its natural solutions using purely arithmetic methods.

The geometric analog of the system is the triangle in the coordinate plane with the sides $y = 25 - 2x$, $y = x$ and $x = 12$ (draw it). Its integer points (inside it and on its sides) are in bijective correspondence with the wanted triangles. One just needs to count them to find how many integer-valued triangles with the sum of the two longest sides being equal to 24 are there.

Alternatively, integer solutions to the system of equations can be counted without the geometric visualization. From the inequalities $25 - 2x \le y \le x$ we have that $25 - 2x \le x$, hence, $x \ge \frac{25}{3}$. Taking into account that x is natural, we conclude that $x \ge 9$. Thus,

$$9 \le x \le 12,$$

and possible values for x are only 9, 10, 11 and 12. If x is one of these numbers, then for y, there are

$$x - (24 - 2x),$$

which is $3x - 24$, possible values. Therefore, overall there are

$$(3 \cdot 9 - 24) + (3 \cdot 10 - 24) + (3 \cdot 11 - 24) + (3 \cdot 12 - 24) = 30$$

natural solutions to the system.

The last, purely arithmetic approach to the calculation of the number of solutions to the system of inequalities lend itself best to generalization, which is the task of the next exercise.

Problem 1.120. *How many integer-valued triangles exist where the sum of the two longest sides equals m?*

Answer. $\left(\left[\frac{m}{2}\right] - \left[\frac{m}{3}\right]\right) \cdot \left(\frac{3}{2}\left[\frac{m}{2}\right] + \frac{3}{2}\left[\frac{m}{23}\right] + \frac{3}{2} - m\right)$; *The symbol* $[a]$ *denotes the integer part of the number a.*

Comment. It may seem unnecessary to solve the general problem if its particular characteristic case has already been solved. However, certain complications appear in the general case, and it is important to learn to deal with them.

Using the same notation as before, for the pairs $(x; y)$, which bijectively correspond to the wanted triangles, we have the following system of inequalities:

$$\begin{cases} \left[\frac{m}{3}\right] < x \le \left[\frac{m}{2}\right], \\ m - 2x < y \le x. \end{cases}$$

Problem 1.121. *How many integer-valued triangles exist, where the sum of two shortest sides equals m?*

Answer. $\frac{1}{2}\left[\frac{m}{2}\right] \cdot \left(\left[\frac{m}{2}\right] + 1\right)$.

Problem 1.122. *How many different integer-valued triangles are there with a perimeter of 30? Find all their sides.*

Answer. 12. *The wanted triangles are:* $(3;13;14)$, $(4;12;14)$, $(5;12;13)$, $(5;11;14)$, $(6;11;13)$, $(6;10;14)$, $(7;11;12)$, $(7;10;13)$, $(7;9;14)$, $(8;10;12)$, $(8;9;13)$, $(9;10;11)$.

Hint. For example, denote the sides of the triangle in the following way: x, y, $30-x-y$, where $x < y < 30-x-y$. The necessary and sufficient conditions for the existence of a triangle are described by the inequality $x+y > 30-x-y$. Hence, there is a bijection between the wanted triangles and natural solutions to the system of inequalities:

$$\begin{cases} x < y, \\ x+2y < 30, \\ x+y > 15. \end{cases}$$

Now, find its analog on the coordinate plane and count that integer points, which are inside the corresponding shape.

Problem 1.123. *How many different equilateral integer-valued triangles are there with a perimeter of n, whose lateral sides are not longer than the base?*

Answer. $\left[\frac{n}{3}\right] - \left[\frac{n}{4}\right]$.

Hint. Let $(x; x; n-2x)$ be the wanted triangle. Then

$$\begin{cases} x \le n-2x, \\ x+x > n-2x. \end{cases}$$

The natural solutions of this system of inequalities are in bijective correspondence with the wanted triangles.

Problem 1.124. *How many different equilateral integer-valued triangles are there with a perimeter of n, whose lateral sides are not shorter than the base?*

Answer. $\left[\frac{n-1}{2}\right] - \left[\frac{n-1}{3}\right]$.

Problem 1.125. *How many different equilateral integer-valued triangles are there, having one of their sides equal to a, and the ratio of two others equal to the ratio of p to q (p and q are different natural numbers $(p > q)$, which do not have common divisors except 1).*

Answer. $\left[\frac{a-1}{p-q}\right] - \left[\frac{a}{p+q}\right]$.

Solution. Let us denote the unknown sides by px and qx. Note that x should be integer. Really, if we suppose that $x = \frac{k}{s}$, where k and s are natural and mutually prime numbers, then $\frac{pk}{s}$ and $\frac{qk}{s}$ are integer, hence, p and q are divisible by s. However, p and q are mutually prime, so $s = 1$.

In order for a triangle with the sides px, qx and a to exist, the inequalities

$$\begin{cases} px+qx > a, \\ qx+a > px, \end{cases}$$

have to hold. They can be joined into equivalent two-sided inequality:

$$\frac{a}{p+q} < x < \frac{a}{p-q}.$$

This establishes a bijection between the wanted integer-valued triangles and integer points from the interval

$$\left(\frac{a}{p+q}, \frac{1}{p-q}\right).$$

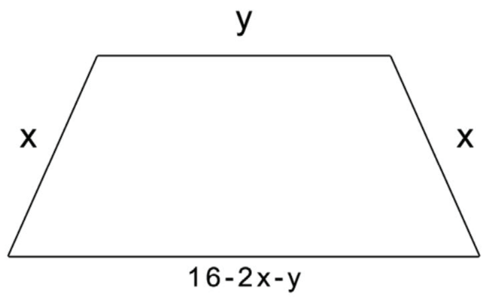

Figure 1.25. Equilateral integer-valued trapezia.

The greatest integer from this interval is $\left[\frac{a-1}{p-q}\right]$, and the smallest one is $\left[\frac{a}{p+q}\right]+1$. Therefore, there are $\left[\frac{a-1}{p-q}\right]-\left[\frac{a}{p+q}\right]$ integer points in the interval.

Problem 1.126. *How many different equilateral integer-valued triangles have a perimeter of n?*

Answer. $\left[\frac{p-1}{2}\right]-\left[\frac{p}{4}\right]$.

Problem 1.127. *How many equilateral integer-valued trapezia have a perimeter of 16?*

Answer. 9.

Solution. Let x be the lateral side of a trapezium and y be its shorter base. Then $16-2x-y$ is its longer base (see Fig. 1.25).

The numbers x, y and $16-2x-y$ can not be chosen arbitrarily. First, the following inequality must hold:

$$y<16-2x-y.$$

In addition, there should be

$$2x+y>16-2x-y$$

(a polygonal chain is longer than the line segment connecting its ends). Finally, x and y are positive numbers (and they are integer by the statement of the problem).

The conditions (inequalities) ensuring the existence of a trapezium are summarized below:

$$\begin{cases} x>0, \\ y>0, \\ x+y<8, \\ 2x+y>8. \end{cases} \tag{1.17}$$

We have a system of linear inequalities which can be interpreted geometrically. Its solutions are the points inside the triangle ABC (see Fig. 1.26) excluding the points on its bounds.

The wanted trapezia correspond to the integer points lying inside the triangle ABC. Overall there are 9 of them. Below is the list of such trapezia (the first two numbers are

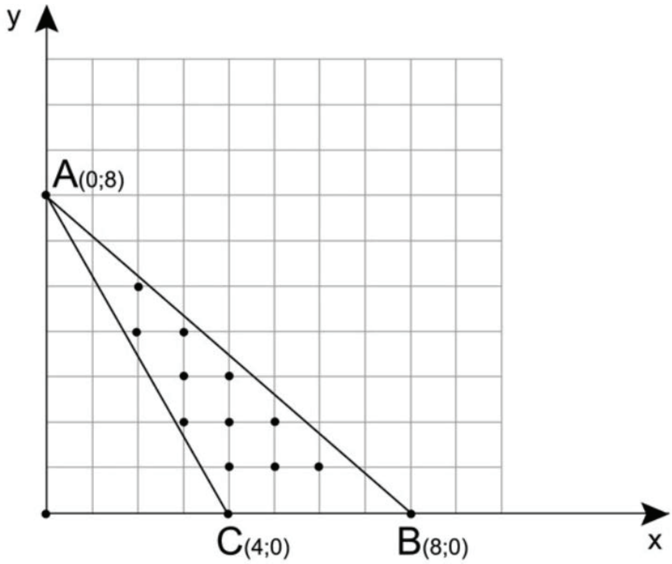

Figure 1.26. Geometric interpretation of a system of linear inequalities.

lateral sides followed by shorter (y) and longer $(16-2x-y)$ bases): $(4;4;1;7)$, $(5;5;1;5)$, $(6;6;1;3)$, $(4;4;2;6)$, $(5;5;2;4)$ $(3;3;3;7)$, $(4;4;3;5)$, $(3;3;4;6)$, $(2;2;5;7)$.

Is there a method of calculation of integer points inside the triangle *ABC*, which can be generalized for the case of an arbitrary perimeter p?

Clearly, the easiest course of action is the following. First, we find the number of integer points inside the triangle *ABC*; next, we find the number of integer points inside the triangle *ACO* and on its side *AC*. The difference between these two numbers is the required number. We apply this approach to the next problem.

Problem 1.128. *How many equilateral integer-valued trapezia have a perimeter of p?*

Solution. The characteristics of the equations of the lateral sides *AB* and *AC* of the triangle *ABC* (see the previous exercise) advice that it is reasonable to consider the cases of even and odd values of p separately.

Let $p=2k$ (k is a natural number). In this case the system of inequalities (1.17) transforms into

$$\begin{cases} x>0,\ y>0, \\ x+y<k, \\ 2x+y>k. \end{cases}$$

The triangle *ABO* is bounded from above by the line $y=k-x$, and the triangle *ACO* is bounded by the line $y=k-2x$.

Integer points inside the triangle *ABO* have abscissas belonging to the interval $[1,k-2]$. Fix one of these values and denote it by x. The integer points with such abscissas from the triangle *ABO* have ordinates from the interval $[1,k-x)$. Hence, there exist $k-x-1$ such points. The amount of all integer points from the triangle *ABO* is given by

$$(k-1-1)+(k-2-1)+(k-3-1)+\ldots+(k-(k-2)-1).$$

The summands here are the values of the expression $k-x-1$ for all possible values of x, that is for values of x from 1 to $k-2$. As the elements of the above sum are members of an arithmetic progression, the sum reduces to

$$\frac{(k-2)(k-1)}{2}.$$

Similarly, we count all integer points inside the triangle ACO and on its side AC (excluding the ends of this side). This time, for every $x \in \left[1,\ \left[\frac{k}{2}\right]\right]$, we have

$$1 \le y \le k-2x,$$

hence, there are $k-2x$ values for y. Therefore, there are

$$\begin{aligned}&(k-2)+(k-4)+(k-6)+\ldots+\left(k-2\left[\tfrac{k}{2}\right]\right)=\\&=k\cdot\left[\tfrac{k}{2}\right]-\left[\tfrac{k}{2}\right]\cdot\left(\left[\tfrac{k}{2}\right]+1\right)=\left[\tfrac{k}{2}\right]\cdot\left(k-\left[\tfrac{k}{2}\right]-1\right)\end{aligned}$$

wanted points. The number of equilateral integer-valued trapezia with the perimeter of $p=2k$ is given by

$$\frac{(k-2)(k-1)}{2}-\left[\frac{k}{2}\right]\cdot\left(k-\left[\frac{k}{2}\right]-1\right). \tag{1.18}$$

This formula does not look attractive. Much easier formulas might be obtained if the case where $p=2k$ is split further into two more cases: odd and even k.

If $k=2s$ (that is, $p=4s$), then $\left[\frac{k}{2}\right]=s$, and unsightly expression (1.18), transforms into compact and beautiful formula

$$(s-1)^2.$$

In the case $k=2s+1$ (that is, $p=4s+2$), the answer also has simple form:

$$s(s-1).$$

II. The amount of equilateral integer-valued trapezia with a perimeter of $p=2k+1$ can be calculated similarly. We suggest the reader make it. The result should be

$$\frac{k(k-1)}{2}-\left[\frac{k}{2}\right]\cdot\left(k-\left[\frac{k}{2}\right]-1\right).$$

When $k=2s$ $(p=4s+1)$, this expression reduces to

$$s^2,$$

and for $k=2s+1$ $(p=4s+3)$ it turns into

$$s\cdot(s+1).$$

Problem 1.129. *There are* 24 *points on a circle splitting it into arcs of the same length. Let us call these points the base points. How many different regular polygons exist with all their vertices lying in the base points?*

Answer. 6.

Problem 1.130. *(Generalization of the previous problem). Let*

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$$

be the prime factorization of the number n (p_1, p_2, ..., p_s are different numbers). There are n base points on a circle splitting it into n arcs of equal length. How many different regular polygons exist with all their vertices lying in the base points?

Answer. $(k_1+1)(k_2+1)\cdots(k_s+1)-1$.

Problem 1.131. *(Continuation of the previous problem). What is the answer to the previous problem if*

$$n = 2^{k_0} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s}$$

(k_0 is natural)?

Answer. $(k_0+1)(k_1+1)(k_2+1)\cdots(k_s+1)-2$.

4. Recurrence

4.1. Sequences

A sequence is an infinite collection of numbers ordered following the example of the natural series, which is itself a prime and benchmark sequence. It has its beginning (the number 1) and has no end. When we count: one, two, three, four, ..., we spell out natural numbers in the order, in which they form the most fundamental of all sequences. The ability to make a further step at any time during the counting evidence the infinite nature of this sequence.

The structure of the sequence of natural numbers (natural series) can be completely described by several definitive properties, which we have been familiar with since the first years of study of arithmetic. These properties are outlined below.

1. The first natural number is 1. This is the only natural number which has no predecessor.
2. For every natural number, there is a successive one, and the successor is unique.
3. Every natural number, except for 1, has a preceding one, and the predecessor is unique.
4. Starting with the number 1, then moving to the next number (2), and to the next (3) and so on, after finite (though possibly large) amount of steps we will get to any natural number.

The last property might be hard to understand but it is extremely important. Actually, it means that although the natural series is infinite, every natural number has finite place in it, if one begins the count at 1.

Now, assume that under every number of the natural series we write another number following some rule (denote these numbers by a_i, and let the index i coincide with the corresponding natural number):

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \dots & n & \dots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_n & \dots \end{array}$$

The numbers a_i are now placed next to each other imitating the elements of the natural series. These numbers are said to form the sequence and are called the elements (members, terms) of this sequence: a_1 is the first element, a_2 is the second, a_3 is the third, and so on. The sequence is denoted by the symbol (a_i).

By its structure, the sequence (a_i) reminds the natural series. This sequence is similar to it, looks like it, and in fact, is created following its example. What is the difference between the sequence and the natural series? The difference is in the elements of the two. In particular, the first member a_1 is not necessarily 1, the second member does not have to be 2, etc. Moreover, its elements ought not to be natural numbers. They can be *arbitrary* numbers (negative, non-integer, non-rational). It is worth mentioning that different by-order elements of a sequence can be the same numbers. In other words, there are no restrictions imposed on the elements of a sequence. What really matters is the rule of construction of the sequence. It should clearly and unambiguously define every element. We provide several examples of sequences below.

1. The sequence of the squares of natural numbers. Its n-th element is the square of the number n. This sequence begins as follows:

$$1, 4, 9, 16, 25, 36, 49, \dots$$

2. The sequence of the inverse natural numbers:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots$$

3. The sequence of decimal approximations of π:

$$3,1; 3,14; 3,141; 3,1415; 3,14159; \dots$$

4. The sequence of digits after the comma in the decimal representation of π:

$$1, 4, 1, 5, 9, 2, 6, 5, 3, 6, \dots$$

5. The (increasing) sequence of prime natural numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \dots$$

6. The sequence of repeating 1 and 0:

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$$

The above examples should provide a clear insight into the variety of numeric sequences. In addition, the evidence that the methods of description of different sequences are impossible to unify or even classify. For instance, the sequence of prime numbers (4) is described by just a couple of words. However, one needs to be aware that this description is nowhere near being elementary because it uses the notion of prime numbers, which in its turn bases on the notion of divisibility, while the latter exploits the definition of a product. In other words, the phrase “sequence of prime numbers” comprehends an essential part of arithmetic.

The sequence of decimal approximations of π is even more complex. In order to understand, what is it about, one requires to be familiar with the idea of the expression of real numbers as infinite decimal fractions. Moreover, as it is necessary to write down actual members of this sequence, one has to be able to calculate them. This task is from the field of high-order math.

The sequences (1), (2) and (6) are created following quite straightforward rules and their elements can be given by compact and transparent computational formulas. The rule for sequence (1) is as follows: in order to find the n-th element of the sequence, one has to square its number n. This is easily expressed by the formula:

$$a_n = n^2,$$

where the symbol a_n underlines that we deal with the n-th by order element of the sequence. Similarly, there are formulas for the sequences (2) and (6). The element of the former are given by

$$a_n = \frac{1}{n},$$

while the formula for the latter is

$$a_n = \frac{1}{2}(1+(-1)^{n-1}).$$

It seems appropriate to call this type of formula a direct formula. They allow calculating every member of a sequence by its number. That is why it is reasonable to use direct formulas: they provide the exact value of any member of a sequence once its number is known. In addition, a direct formula provides an opportunity to answer a variety of questions concerning the global characteristics of a sequence and not only individual members of a sequence. In particular, the former includes the answers to the following questions: what is the set of values of members of the sequence; is the sequence increasing (each next member is greater than the previous one); is the sequence decreasing; is there the largest (smallest) member in the sequence, and if so, what exactly; how do the members of the sequence behave with the growth of their numbers (do they increase to infinity or decrease, do they tend to get closer to some number, etc.) and so on.

Below we provide other examples of formulas defining certain sequences.

1. $a_n = \frac{1+n}{1+n^2}$. The sequence having this direct computational formula begins as follows:

$$1, \frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, \frac{4}{25}, \frac{9}{65}, \dots$$

2. $a_n = \sin\frac{n\pi}{2}$. Here are several starting members of the sequence defined by this formula:

$$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

The properties of the function $\sin x$ suggest that this sequence is periodic. Four of its members $1, 0, -1, 0$ repeat cyclically.

3. $a_n = n + (-1)^n$. The starting members are:

$$0, 3, 2, 5, 4, 7, 6, 9, 8, \ldots$$

In order for the expression $f(n)$ to define some sequence it is necessary and sufficient that it can be calculated for any natural value of the variable (argument) n. Under this condition the formula $a_n = f(n)$ defines the sequence

$$f(1),\ f(2),\ f(3), \ldots,\ f(n), \ldots$$

As we can see, the idea of the definition of a sequence with a direct computational formula is in essence no different from the more general idea of the definition of an arbitrary real function with a computational formula. The difference is insignificant. It is only in the fact that in the case of an arbitrary real function its domain (the set of values of the argument) can vary in a very broad range, and in the case of a sequence, it necessarily coincides with the set of natural numbers. A sequence is a function of a natural argument.

4.2. Definition of a Sequence by a Recurrence Relation

The fact that a sequence is a function of natural argument, and its members are ordered as a natural series, there is another opportunity to define it, which is essentially different from the previous. In the above discussion, we have considered the direct rule of dependence of the members of a sequence on their numbers. A direct formula explicitly expresses this dependence establishing the correspondence between natural numbers (the numbers of the members of a sequence) and the elements of a sequence.

Another approach is to define the value of each following member of a sequence through values of several previous members and not only with its number. A formula establishing the required relation is called a recurrence relation. An elementary example of such a formula is

$$a_n = a_{n-1} + a_{n-2}.$$

What is the sense of this expression? It tells us about the sequence (a_n), the members of which follow the rule: each of them (as n is an arbitrary natural number) is the sum of two previous members (because a_{n-1} and a_{n-2} immediately precede a_n). Is this information about the sequence sufficient to reproduce it? For instance, are we able to determine a few of its starting elements? Clearly, the answer is no. In particular, it is impossible to determine the first member of the sequence. As well as the second one. The formula $a_n = a_{n-1} + a_{n-2}$ can not be applied to the first two members of the sequence, since neither of them has two preceding elements. Therefore, the formula fails from the very beginning. In order to

make it work, it is necessary to define two starting members of the sequence. Given this preliminary information, the formula begins operation, tirelessly and relentlessly expanding the sequence: the third term is the sum of the first and second, the fourth term is the sum of the second and third, etc., to infinity.

Obviously, a recurrence relation defines a *class of sequences* and not the exact sequence. The class comprises all the sequences following this recurrence relation. To distinguish one of the sequences of the class one needs to define a certain amount of its starting members.

Example 1.22. *Some of the sequences defined by the recurrence relation* $a_n = a_{n-1} + a_{n-2}$ *are:*

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \ldots$$
$$-1, 4, 3, 7, 10, 17, 27, 44, 71, \ldots$$
$$\frac{1}{2}, 2, \frac{5}{2}, \frac{9}{2}, 7, \frac{23}{2}, \frac{37}{2}, 30, \frac{97}{2}, \ldots$$

The first of the above is the Fibonacci sequence (after the Italian mathematician of XII-XIII centuries).

Example 1.23. *The sequences defined by the recurrence relation*

$$a_n = a_{n-1} + d$$

(d is fixed nonzero) are called the arithmetic progressions with a common difference of d. This time the recursive formula becomes usable upon the definition of the first member of a sequence. Therefore, every arithmetic progression is defined by two numbers: the first member and the common difference.

Example 1.24. *The recurrence relation*

$$a_n = a_{n-1} \cdot q$$

$(q \neq 1)$ *with the initial member* $a_1 \neq 0$ *define a sequence known as a geometric progression. The number q is called the common ratio. For example,*

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

is a geometric progression with the initial member 1 and the common ratio 2. It follows the recursive formula $a_n = 2a_{n-1}$ *(each consecutive member is twice the previous one).*

Example 1.25. *The recurrence relation (recursive formula)*

$$a_n = a_{n-1} - a_{n-2}$$

along with the initial conditions

$$a_1 = 1,\ a_2 = 3$$

define the following periodic sequence:

$$1, 3, 2, -1, -3, -2, 1, 3, 2, \ldots$$

with the period $(1, 3, 2, -1, -3, -2)$ *of 6.*

Example 1.26. *The recurrence relation*

$$a_n = \frac{a_{n-1} \cdot a_{n-2}}{a_{n-3}}$$

expresses each subsequent term of a sequence with three preceding terms. The element with the smallest number that can be determined using this formula is a_4. Thus, three initial terms of a sequence have to be defined. If we put

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,$$

then we get the following sequence:

$$1, 1, 2, 2, 4, 4, 8, 8, 16, 16, ...$$

Another choice of the first elements leads to completely different sequence. For example, setting

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3,$$

we get another sequence obeying the given recurrence relation:

$$1, 2, 3, 6, 9, 18, 27, 54, 81, \ldots$$

Let us summarize the findings of this section. A recurrence relation defines a class of sequences instead of some exact sequence. Preliminary information, which we call initial conditions, is required to distinguish a certain sequence from the class. These conditions are nothing but the exact values of those initial terms of a sequence, to which the recurrence relation can not be applied. The number of such terms depends on the type of a recurrence relation and is completely defined by it.

4.3. Relation between Recursive and Direct Formulas

Is it possible to define the same sequence by formulas of two types: by a recursive formula and a direct one? There is no exact answer to this question in a general setting. It depends on the range of methods allowed for the construction of the formulas of both types. Having no intention to give a comprehensive answer, we provide some sensible recommendations to find the answer to this reasonable question in important practical cases.

First, we consider the transition from a direct formula to a recursive one. This transition is always possible, though there is not much sense in it as it can be performed in infinitely many ways. There is a simple example that illustrates this. Let us have the sequence

$$a_n = 2^{n-1}.$$

This is a geometric progression with the common ratio of 2 and the initial term 1.

Below, there are several transformations of this direct formula into a recursive one:

1. $a_n - a_{n-1} = 2^{n-1} - 2^{n-2} = 2^{n-2}$, hence

$$a_n = a_{n-1} + 2^{n-1};$$

2. $a_n \cdot a_{n-1} = 2^{n-1} \cdot 2^{n-2} = 2^{2n-3}$, which yields

$$a_n = \frac{2^{2n-3}}{a_{n-1}};$$

3. $\frac{a_n}{a_{n-1}} = \frac{2^{n-1}}{2^{n-2}} = 2$, hence

$$a_n = 2a_{n-1}.$$

Thus, the sequence defined by the direct formula $a_n = 2^{n-1}$ can also be defined by recurrence relations:

$$a_1 = 1,\ a_n = a_{n-1} + 2^{n-1},$$

or

$$a_1 = 1,\ a_n = \frac{2^{2n-3}}{a_{n-1}},$$

or

$$a_1 = 1,\ a_n = 2 \cdot a_{n-1},$$

or by infinite number of others.

Serious problems can be encountered while attempting the reverse transition from a recursive formula to a direct one. In fact, such a transition is not always possible. And when it actually is, performing it requires more than just technical exercise. In most cases, the success of transition is down to the combination of erudition, creativity, and luck.

One of the possible approaches to the search, design, and validation of a direct formula can be described in general terms as follows. Having written down several initial members of the sequence (based on the recursive formula and initial conditions), we try to "tie" them directly to their numbers and guess the direct formula. If we are lucky enough to come up with the idea of a direct formula, then it is necessary to prove it, to make sure that it has been guessed correctly. This stage is much simpler than the first one. While in the first stage (it is appropriate to call it heuristic) the decisive role is played by ingenuity, analytical abilities and experience of the researcher, on the second stage it is down to purely technical skills, as only two things are remaining to check: first, whether the hypothetical formula defines correctly those elements of the sequence which are prescribed by the initial conditions; and secondly, is the recursive formula valid for it?

Example 1.27. *The direct formula for an arithmetic progression*

$$a_1 = a;\ a_n = a_{n-1} + d$$

is

$$a_n = a + d(n-1),$$

as for $n = 1$ it produces the correct value $a_1 = a$, and the recursive formula is correct for it as

$$a_n - a = d(n-1)$$

and

$$a_{n-1} + d = a + d(n-2) + d = a + d(n-1).$$

Example 1.28. *The direct formula for a geometric progression*

$$a_1 = a,\ a_n = a_{n-1} \cdot q$$

is

$$a_n = aq^{n-1}.$$

Example 1.29. *Let the sequence be defined recursively by*

$$a_1 = 2,\ a_n = 3a_{n-1} + 2. \tag{1.19}$$

Let us try guessing its direct formula. We deduce several (five) initial terms of the sequence:

$$2, 8, 26, 80, 242, ... \tag{1.20}$$

Having thoroughly investigated the above numbers we notice that all of them are less by 1 than the powers of number 3. Here are the consecutive powers of 3:

$$3, 9, 27, 81, 243, \ldots$$

The corresponding elements of the sequence are less by 1. Hypothesis: the direct formula of sequence (1.22) is

$$a_n = 3^n - 1. \tag{1.21}$$

Why do we call it a hypothesis? Why can't we consider the developed formula to be truly the direct one for our sequence? Because it is based on rather limited information about the sequence it tries to resemble. Constructing the formula we have taken into account only five members of the sequence. Therefore, we need to provide additional rationale to back this formula. It will be completely proved only if it appears that the sequence defined by the recurrence relation (1.19) and the sequence defined by the direct formula (1.21) are the same. Thus, we need to find out whether relationship (1.19) holds for formula (function) (1.21). We have:

1. $a_1 = 3^1 - 1 = 2$;
2. $a_n = 3^n - 1$; $3a_{n-1} + 2 = 3(3^{n-1} - 1) + 2 = (3^n - 3) + 2 = 3^n - 1$.

Therefore,

$$a_n = 3a_{n-1} + 2$$

and $a_n = 3^n - 1$. Our guess of the direct formula is correct.

We attempt to find the direct formula for the sequence defined by the following conditions for its first two members and the recursive formula:

$$a_1 = 2,\ a_2 = 3,\ a_n = 3a_{n-1} - 2a_{n-2}.$$

To this end, we write down the first six members of the sequence:

$$2, 3, 5, 9, 17, 33, \ldots$$

We decrease the above numbers by 1 to get the sequence the rule of construction of which is unquestionable:

$$1, 2, 4, 8, 16, 32, \ldots$$

These are the consecutive powers of 2 (beginning with power 0). We make the assumption that the direct formula for our sequence is

$$a_n = 2^{n-1} + 1.$$

It remains to provide an appropriate proof. For $n = 1$ and $n = 2$ this formula correctly produces two initial terms of the sequence: $a_1 = 2$, $a_2 = 3$. Further, we have

$$a_n = 2^{n-1} + 1,\ a_{n-1} = 2^{n-2} + 1,\ a_{n-2} = 2^{n-3} + 1;$$

$$\begin{aligned} 3a_{n-1} - 2a_{n-2} &= 3(2^{n-2} + 1) - 2(2^{n-3} + 1) = \\ &= 3 \cdot 2^{n-2} + 3 - 2^{n-2} = 2^{n-1} + 1 = a_n. \end{aligned}$$

The original recurrence relation holds true for the found direct formula. Hence, the direct formula is determined correctly.

Example 1.30. *Let a sequence be defined by its two initial terms*

$$a_1 = x + y,$$

$$a_2 = \frac{x^3 - y^3}{x - y}$$

(x and y are given non-equal numbers) and the recurrence relation

$$a_n = (x + y)a_{n-1} - xya_{n-2}.$$

What is the direct formula for this sequence?

Again, we begin with an experiment: we calculate several first terms of the sequence one be one, attempting to guess their dependence on their numbers. We have:

$$\begin{aligned} a_3 &= (x+y)a_2 - xya_1 = (x+y)\tfrac{x^3-y^3}{x-y} - xy(x+y) = \\ &= (x+y)\tfrac{x^3-y^3-x^2y+xy^2}{x-y} = (x+y)\tfrac{x^2(x-y)+y^2(x-y)}{x-y} = \\ &= \tfrac{x^4-y^4}{x-y}; \end{aligned}$$

$$\begin{aligned} a_4 &= (x+y)a_3 - xya_2 = (x+y)\tfrac{x^4-y^4}{x-y} - xy\tfrac{x^3-y^3}{x-y} = \\ &= \tfrac{x^5-xy^4+x^4y-y^5-x^4y+xy^4}{x-y} = \tfrac{x^5-y^5}{x-y}. \end{aligned}$$

The rule defining the expansion of the sequence appears to be obvious now. Let us line up the discovered terms of the sequence, transforming the first term into the form similar to the others:

$$\frac{x^2 - y^2}{x - y},\ \frac{x^3 - y^3}{x - y},\ \frac{x^4 - y^4}{x - y},\ \frac{x^5 - y^5}{x - y},\ \ldots.$$

This is the case when it is impossible to fail guessing the direct formula:

$$a_n = \frac{x^{n+1} - y^{n+1}}{x - y}.$$

It remains to expose it to strict and decisive check by the recursive formula. We have:

$$\begin{aligned}
&a_{n-1} = \tfrac{x^n - y^n}{x-y},\ a_{n-2} = \tfrac{x^{n-1} - y^{n-1}}{x-y};\\
&(x+y)a_{n-1} - xya_{n-2} = (x+y)\tfrac{x^n - y^n}{x-y} - xy\tfrac{x^{n-1} - y^{n-1}}{x-y} =\\
&= \tfrac{x^{n+1} - y^{n+1}}{x-y} = a_n.
\end{aligned}$$

The candidate for the direct formula has successfully passed the check.

Sometimes one can derive the direct formula for a sequence defined by a recursive one using the “descent approach”. Below there are two examples explaining its essence.

Example 1.31. *A sequence is defined by the recursive formula*

$$a_n = a_{n-1} + (2n-1)$$

and the initial term $a_1 = 1$.

In order to find the direct formula we write down all the equalities which can be derived by application of the recurrence relation to all the terms of the sequence from a_n to a_2, ending with the equality $a_1 = 1$:

$$\begin{aligned}
&a_n = a_{n-1} + (2n-1)\\
&a_{n-1} = a_{n-2} + (2n-3)\\
&a_{n-2} = a_{n-3} + (2n-5)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_3 = a_2 + 5\\
&a_2 = a_1 + 3\\
&a_1 = 1.
\end{aligned}$$

Now, sum up the left-hand and right-hand sides of these equalities. Having excluded the same terms from the both sides we get:

$$a_n = 1 + 3 + 5 + \ldots + (2n-5) + (2n-3) + (2n-1).$$

The right-hand side is the sum of n members of arithmetic progression which can be reduced by the well-known formula. Thus, we have derived the desired direct formula for the given sequence:

$$a_n = n^2.$$

Example 1.32. *Let a sequence be defined by the recursive formula*

$$a_n = \frac{n-1}{n} a_{n-1}$$

and the initial condition $a_1 = 1$.

Following the path of the previous example we create a string of formulas:

$$\begin{aligned}
&a_n = \tfrac{n-1}{n}a_{n-1}\\
&a_{n-1} = \tfrac{n-2}{n-1}a_{n-2}\\
&a_{n-2} = \tfrac{n-3}{n-2}a_{n-3}\\
&\dots\dots\dots\dots\dots\dots\dots\\
&a_3 = \tfrac{2}{3}a_2\\
&a_2 = \tfrac{1}{2}a_1\\
&a_1 = 1.
\end{aligned}$$

Now, we find the products of the right-hand and left-hand sides of the above equalities and divide the resulting equality by $a_{n-1}a_{n-2}\ldots a_1$. The direct formula for our sequence follows:

$$a_n = \frac{1}{n}.$$

For certain types of recurrence relations, there are particular approaches to the transition from recursive formulas to direct ones, based on the idea of "simplification" of recurrence relations. In this regard, we give two examples, which refer to rather common (in particular, in combinatorial problems) recurrences.

Example 1.33. *Let a sequence be defined by its initial term*

$$a_1 = a$$

and the recursive formula

$$a_n = pa_{n-1} + d \tag{1.22}$$

Is there a direct formula for this sequence?

If $p = 1$, then the above recurrence relation defines an arithmetic progression with the common difference d, for which the direct formula is known. There are no problems when $d = 0$ as well because in this case, the recursive formula defines a geometric progression with the common ratio p. The direct formula of such progression is also readily available.

Hence, we find ourselves in an unfamiliar situation when $p \neq 1$ and $d \neq 0$. Assuming these conditions hold, rearrange the recursive formula in the following way:

$$a_n - x = y(a_{n-1} - x), \tag{1.23}$$

where x and y are unknown. Is it possible to choose such values for the unknowns that equality (1.23) coincides with equality (1.22)? We express equality (1.23) in the form (1.22):

$$a_n = ya_{n-1} + x - xy.$$

Comparing this equality with equality (1.22), we see that we can put

$$y = p,\ x - xy = d,$$

Hence, $x = \frac{d}{1-p}$.

Thus, the recurrence relation (1.22) transforms into the following relation

$$a_n - \frac{d}{1-p} = p\left(a_{n-1} - \frac{d}{1-p}\right) \tag{1.24}$$

What is the purpose of doing so? We illustrate the answer by the formula with the help of the following notation:

$$c_n = a_n - \frac{d}{1-p}. \tag{1.25}$$

Really, now the formula (1.24) transforms into

$$c_n = pc_{n-1}.$$

The above equality evidences that the sequence (c_n) is a geometric progression with the common ratio p. Taking into account that

$$c_1 = a_1 - \frac{d}{1-p} = a - \frac{d}{1-p},$$

we get the direct formula for this sequence:

$$c_n = \left(a - \frac{d}{1-p}\right)p^{n-1}.$$

Equality (1.25) expresses the relation between the geometric progression (c_n) and the original sequence (a_n). Having obtained the direct formula for the progression (c_n), we determine the direct formula of the sequence (a_n) from (1.25):

$$a_n = \left(a - \frac{d}{1-p}\right)p^{n-1} + \frac{d}{1-p}.$$

Example 1.34. *Let a sequence be defined by its initial terms*

$$a_1 = a,\ a_2 = b$$

and the recurrence relation

$$a_n = (x+y)a_{n-1} - xya_{n-2}, \tag{1.26}$$

where x and y are given non-equal numbers.

The task is to determine the direct formula for this sequence.

There is an important peculiarity in this problem. Because of the fact that a, b, x and y are arbitrary numbers (satisfying the only condition $x \neq y$), we are not dealing here with any certain sequence (a_n), but with a wide class of similar (being of the same type) sequences, related by the algebraic form of the recurrence rules, which define them. By the way, the same situation was in the previous example.

We express the recursive formula (1.26) in the form

$$a_n - ya_{n-1} = x(a_{n-1} - ya_{n-2}). \tag{1.27}$$

Denoting $a_n - ya_{n-1}$ by b_{n-1}, we can see that the sequence (b_{n-1}) is a geometric progression, as equality (1.27) yields that

$$b_{n-1} = xb_{n-2}.$$

Taking into account that $b_1 = a_2 - ya_1 = b - ya$, we get the direct formula for b_{n-1}:

$$b_{n-1} = (b - ya) \cdot x^{n-2}. \tag{1.28}$$

Alternatively, recurrence relation (1.19) can be expressed as

$$a_n - xa_{n-1} = y(a_{n-1} - xa_{n-2}). \tag{1.29}$$

Denoting $a_n - xa_{n-1}$ by c_{n-1}, we get the recursive formula

$$c_{n-1} = yc_{n-2}$$

for the sequence (c_{n-1}). Additionally, from the equalities $a_1 = a$ and $a_2 = b$ we conclude that

$$c_1 = a_2 - xa = b - xa.$$

Hence, the sequence (c_{n-1}) obeys the following direct formula:

$$c_{n-1} = (b - xa)y^{n-2}. \tag{1.30}$$

There are two equalities for a_n and a_{n-1}:

$$\begin{aligned} a_n - ya_{n-1} &= (b - ya)x^{n-2}, \\ a_n - xa_{n-1} &= (b - xa)y^{n-2}. \end{aligned}$$

Excluding a_{n-1} from them, we get the direct formula for a_n. Technically, the procedure of exclusion can be carried out as follows: multiply the first equality by x, and the second by y, and then subtract the second equality from the first term-wise. We get:

$$(x - y)a_n = (b - ya)x^{n-1} - (b - xa)y^{n-1}.$$

Taking into account that $x \neq y$, we derive a_n from the above equality:

$$a_n = b \cdot \frac{x^{n-1} - y^{n-1}}{x - y} - axy \cdot \frac{x^{n-2} - y^{n-2}}{x - y}.$$

There is one question remaining: what will happen if in the recurrence relation (1.26) the numbers x and y are the same? Indeed, in this case equations (1.27) and (1.29) are the same and a_{n-1} can not be excluded as above. However, the direct formula can be deduced in this situation as well. Moreover, it can be performed in various ways. We present one of them below.

So now we are dealing with the sequence defined by the initial conditions

$$a_1 = a, \; a_2 = b,$$

and the recurrence relation

$$a_n = 2xa_{n-1} - x^2 a_{n-2}.$$

Rearranging the above as

$$a_n - xa_{n-1} = x(a_{n-1} - xa_{n-2}),$$

we find that the sequence $(a_n - xa_{n-1})$ is a geometric progression with the common ratio x and the initial term $b - xa$. Hence,

$$a_n - xa_{n-1} = (b - xa)x^{n-2},$$

and

$$a_n = xa_{n-1} + (b - xa)x^{n-2}.$$

Replicating this formula for all indices less than n we get the following set of formulas:

$$\begin{array}{l|l}
a_n = xa_{n-1} + (b-xa)x^{n-2}, & \\
a_{n-1} = xa_{n-2} + (b-xa)x^{n-3}, & x \\
a_{n-2} = xa_{n-3} + (b-xa)x^{n-4}, & x^2 \\
\dots\dots\dots\dots\dots\dots\dots\dots & \\
a_4 = xa_3 + (b-xa)x^2, & x^{n-4} \\
a_3 = xa_2 + (b-xa)x, & x^{n-3} \\
a_2 = xa_1 + (b-xa), & x^{n-2} \\
a_1 = a, & x^{n-1}
\end{array}.$$

Multiply the second of these equalities by x, the third by x^2, the fourth by x^3 and so on. After this we sum up all the equalities (the left-hand sides and right-hand sides separately). Upon the exclusion of the same terms appearing in the both sides of the resulting equality we get the direct formula for a_n:

$$a_n = ax^{n-1} + (n-1)9b - xa)x^{n-1}.$$

We have completely determined the direct formula for the sequence defined by the initial conditions

$$a_1 = a,\ a_2 = b,$$

and the recurrence relation

$$a_n = (x+y)a_{n-1} - xya_{n-2}. \tag{1.31}$$

How wide is the class of sequences defined by this rule? Visually, this recurrence relation is a special case of a linear homogeneous binomial recurrence relation

$$a_n = pa_{n-1} + qa_{n-2} \tag{1.32}$$

with given p and q. However, formulas (1.31) and (1.32) are equivalent in the sense that every recurrence relation (1.32) can be expressed in the form (1.31). Really, let the values of p and q be given. Are there such x and y that the equalities

$$x + y = p, \quad xy = -q$$

hold simultaneously? In other words, does the system of equations

$$\begin{cases} x+y=p, \\ xy=-q \end{cases}$$

always have a solution? Its solutions are the pairs $(x;y)$, components of which are the roots of the equation

$$z^2 - pz - q = 0.$$

A quadratic equation always has one or two (complex) roots. Hence, recurrence relation (1.32) can always be expressed in the form (1.31). If p and q are real numbers and we require x and y to be real as well, the recurrence relation (1.32) can be expressed in the form (1.31) only if $p^2 + 4q \geq 0$.

4.4. Recurrence Relations in Combinatorial Problems

Is there any relation between sequences and combinatorial problems? Yes, there is. Moreover, sequences appear in combinatorial problems mostly in the context of recurrence relations.

Example 1.35. *There is a path leading to a rabbit hole. The path is a line of squares. Walking on this path a rabbit jumps into the nearest square or one square further, randomly choosing from these two options. How many ways are there for the rabbit to reach the n-th square?*

In order to solve this problem, we need to define a formula (direct or recursive) of a certain sequence. Which sequence is that? And how is this sequence related to the problem?

Denote the sought amount in any appropriate way, say, by γ_n. The index n is not only appropriate here but even necessary as the answer should depend on n. Having answered the question of the problem, that is, having determined the amount of ways for the rabbit to reach the n-th square, we will find the answer to an infinite amount of questions concerning the exact values of n: $1, 2, 3, 4, \ldots$. Having the formula for arbitrary n, we will know γ_1, and γ_2, and γ_3, and so on. In other words, we will know the law of expansion of the sequence (γ_n), and thus will be able to calculate every element (at least potentially). Therefore, although the question seems to be posed in respect of one number, it actually requires us to find the law of expansion of a certain numeric sequence. The sequence, the n-th element of which denotes the number of ways, in which the rabbit can reach the n-th square.

How can this problem be addressed? What could we start with? First, we must clearly understand the situation: what is known and what is to be found. Our aim is clear: we need to guess the law of a certain numerical sequence. What do we know about this sequence? What does the statement of the problem tell us about it? Obviously, the statement of the problem describes the law of the sequence. It appears to be nonsense: we need to find a rule, which is known from the very beginning. However, at the beginning of the problem and in the question we encounter essentially different laws of expansion of the sequence (γ_n). In the statement of the problem, there is a purely descriptive characterization of the sequence. Relying solely on this characterization it is very hard to determine, say, γ_{20}. And the task is to discover the quantitative law of the sequence building upon the qualitative

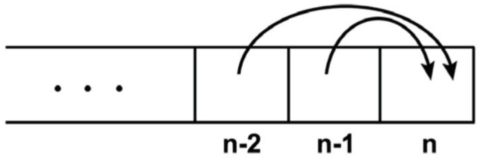

Figure 1.27. Rabbit's way.

description. We have nothing to begin with, except for the aggregation of actual data about the sequence. We directly calculate (thoroughly considering different options) several initial members. The sequence begins as follows

$$1, 2, 3, 5, 8, \ldots$$

(there are three ways to get to the third square, five ways to get to the fourth, eight ways to get to the fifth, and so on).

Even on such a short interval, it is easy to notice a quite simple pattern: each consecutive member of the sequence is the sum of two preceding members:

$$3 = 1 + 2, \; 5 = 2 + 3, \; 8 = 3 + 5.$$

Is this law applicable to all subsequent elements of the sequence? We do not know that. Having considered only five members of the sequence, one can hardly hope to reveal all its secrets. If we were restricted to three initial members, we could have developed a fairly different view on the law of its expansion. However, the observed pattern is an important achievement. Our confidence in it will increase even more if after exploring all the options, we find how many ways are there for the rabbit to get to the sixth square. Because the answer will be 13. Thus, we have a very encouraging hypothesis:

The sequence follows the recurrence relation

$$\gamma_n = \gamma_{n-1} + \gamma_{n-2}.$$

It remains to prove or refute this hypothesis. If it is correct, then the problem is solved. Otherwise, we will have to start from the very beginning.

In order to find why the equality $\gamma_n = \gamma_{n-1} + \gamma_{n-2}$ holds true for any n greater than 2, consider the n-th square on the rabbit's way. What should be the last jump of the string of jumps beginning at the doorstep of the rabbit hole and leading to the n-th square? There are two such jumps: the short one from the $(n-1)$-th square, or the long one from the $(n-2)$-th square. According to the statement of the problem, there are no other options.

So in order to get to the n-th square, the rabbit needs to get to the $(n-2)$-th square and finish his way with a long jump or jump all the way to the $(n-1)$-th square and then jump to the adjacent square. Taking into account that there are γ_{n-2} and γ_{n-1} (according to introduced above notation) ways to get to the $(n-2)$-th and $(n-1)$-th squares respectively, we arrive at the equality

$$\gamma_n = \gamma_{n-1} + \gamma_{n-2}.$$

We have proved the hypothesis and solved the problem. The sought sequence (γ_n) is defined by the initial conditions

$$\gamma_1 = 1,\ \gamma_2 = 2$$

and the recurrence relation $\gamma_n = \gamma_{n-1} + \gamma_{n-2}$. Recalling the formula derived in the previous section or repeating its derivation for our recurrence relation, we can find the direct formula for the sequence. However, it will not make a pleasant impression being too complex.

Example 1.36. *How many ways are there for the rabbit (see the previous problem) to get to the n-th square if he never makes two or more long jumps in a row?*

Denote the sought number by β_n. The n-the square can be reached from the $(n-1)$-th or $(n-2)$-th squares. Count those chains of jumps leading to the n-th square that end with short and long jumps separately. There are β_{n-1} chains of the first type, as this is the number of ways to get to the $(n-1)$-th square, and the rabbit always can make a short jump from it. The situation is different with the $(n-2)$-th square. If the chain of jumps to the n-th square ends with a long jump from the $(n-2)$-th square, then it should be preceded by a short jump from the $(n-3)$-the square, and the chain of jumps up to the $(n-3)$-th square may be arbitrary. Hence, there are β_{n-3} chains of the second type. Thus, we have the recurrence relation

$$\beta_n = \beta_{n-1} + \beta_{n-3}.$$

It remains to accompany it with the initial conditions:

$$\beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 3.$$

Problems

Problem 1.132. *All elements with odd indices of the sequence (c_n) equal -1, and all its elements with even indices are equal to 1. Find the direct formula of the sequence (c_n). Suggest two recurrence relations defining the sequence (c_n).*

Answer. $c_n = (-1)^n$; $c_1 = -1,\ c_n = -c_{n-1}$; $c_1 = -1,\ c_2 = 1,\ c_n = c_{n-2}$.

Problem 1.133. *All elements with odd indices of the sequence (c_n) equal 1, and all its elements with even indices are equal to 2. Find the direct formula of the sequence (c_n). Suggest two recurrence relations defining the sequence (c_n).*

Answer. $a_n = \frac{1}{2}(3+(-1)^n)$; $a_1 = 1,\ a_2 = 2,\ a_n = a_{n-2}$; $a_1 = 1,\ a_n = \frac{3+(-1)^n}{3+(-1)^{n-1}} a_{n-1}$.

Problem 1.134. *Determine six initial terms and the direct formula of a sequence, defined by the recurrence relation and the initial term as follows:*

$$a_1 = 1,\ a_n = -a_{n-1} + 4.$$

Answer. $a_n = 2 + (-1)^n$.

Problem 1.135. *Find the direct formula of the sequence (c_n) defined recursively by*

$$c_1 = 1,\ c_n = 3c_{n-1} + 2.$$

Answer. $c_n = 2 \cdot 3^{n-1} - 1$.

Problem 1.136. *Determine six initial terms and the direct formula of a sequence, defined by the recurrence relation and the initial term as follows:*

$$a_1 = 1,\ a_n = -2a_{n-1} + 5.$$

Answer. $a_n = \frac{(-2)^n+5}{3}$.

Problem 1.137. *Find the direct formulas of sequences defined recursively as follows:*

1. $a_1 = 1,\ a_2 = 1,\ a_n = 7a_{n-1} - 10a_{n-2}$;
2. $a_1 = 1,\ a_2 = 2,\ a_n = 4a_{n-1} - 3a_{n-2}$;
3. $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$.

Answer.

1. $a_n = \frac{2^{n+1}-5^{n-1}}{3}$;
2. $a_n = \frac{3^{n-1}+1}{2}$;
3. $a_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$.

Problem 1.138. *Determine eight initial terms and the direct formulas of sequences defined recursively as follows:*

1. $a_n = \frac{n-1}{n}a_{n-1};\ a_1 = 1.$
2. $a_n = n^2 a_{n-1};\ a_1 = 1.$
3. $a_n = \frac{n}{a_{n-1}};\ a_1 = 1.$
4. $a_n = \frac{a_{n-1}+a_{n-2}}{2};\ a_1 = x,\ a_2 = y.$
5. $a_n = 3a_{n-1} - 2a_{n-2};\ a_1 = 2,\ a_2 = 3.$
6. $a_n = a_{n-1} + \frac{n(n-1)}{2};\ a_1 = 1.$
7. $a_n = 2a_{n-1} - a_{n-2};\ a_1 = 0,\ a_2 = 1.$

Answer.

1. $a_n = \frac{1}{n}$;
2. $a_n = (n!)^2$;
3. $a_n = \begin{cases} \frac{2\cdot 4\cdot 6\cdot\ldots\cdot(2k)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)}, \text{ for } n = 2k; \\ \\ \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2k+1)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2k)}, \text{ for } n = 2k+1; \end{cases}$
4. $a_n = (-1)^{n-1}\cdot\frac{x-y}{3\cdot 2^{n-2}} + \frac{x+2y}{3}$;

5. $a_n = 2^{n-1} + 1$;

6. $a_n = 1 + \frac{n(n^2-1)}{6}$;

7. $a_n = n - 1$.

Problem 1.139. *The symbol* $[x]$ *denotes the integer part of the number x, that is the greatest integer smaller or equal to x (e.g.,* $[2] = 2,\ \left[\frac{5}{2}\right] = 2,\ [\pi] = 3,\ [-\pi] = -4$*).*

Using this notation, define the following sequences with their direct formulas:

1. $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$

2. $1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, \ldots$

3. $-1, -1, 11, -1, -1, 1, 1, -1, -1, \ldots$

4. $0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots$

5. $-1, -1, -1, 1, 1, 1, -1, -1, -1, 1, 1, 1, \ldots$

6. $0, 0, 0, 3, 3, 3, 0, 0, 0, 3, 3, 3, \ldots$

7. $0, 0, 3, 3, 2, 2, 5, 5, 4, 4, 7, 7, \ldots$

Answer.

1. $a_n = \left[\frac{n+1}{2}\right]$;

2. $a_n = \left[\frac{n+2}{3}\right]$;

3. $a_n = (-1)\left[\frac{n+1}{2}\right]$;

4. $a_n = \frac{1}{2}\left((-1)^{\left[\frac{n+1}{2}\right]} + 1\right)$;

5. $a_n = (-1)^{\left[\frac{n+2}{3}\right]}$;

6. $a_n = \frac{3}{2}\left((-1)^{\left[\frac{n+2}{3}\right]} + 1\right)$; $a_n = \left[\frac{n+1}{2}\right] + (-1)^{\left[\frac{n+1}{2}\right]}$.

7. $a_n = \left[\frac{n+1}{2}\right] + (-1)^{\left[\frac{n+1}{2}\right]}$;

Problem 1.140. *Find the direct formula of a sequences defined recursively:*

$$a_1 = 1;\ a_n = n - a_{n-1}.$$

Problem 1.141. *Determine the direct formulas of sequences defined recursively:*

1. $a_1 = 1,\ a_n = a_{n-1} + (2n-1)$;

2. $a_1 = 2,\ a_n = a_{n-1} + 2n$;

3. $a_1 = 1,\ a_n = a_{n-1} + n^2$;

4. $a_1 = 2,\ a_n = a_{n-1} + n(n+1)$;

5. $a_1 = 4,\ a_n = a_{n-1} + 4n^2$;

6. $a_1 = \frac{1}{2},\ a_n = a_{n-1} + \frac{1}{n(n+1)}$;

7. $a_1 = \frac{1}{2},\ a_n = a_{n-1} + \frac{2n-1}{2^n}$;

8. $a_1 = 1,\ a_n = a_{n-1} + n \cdot n!$;

9. $a_1 = \frac{1}{2},\ a_n = a_{n-1} + \frac{n}{(n+1)!}$.

Answer.
1. $a_n = n^2$;
2. $a_n = n(n+1)$;
3. $a_n = \frac{n(n+1)(2n+1)}{6}$.
4. $a_n = \frac{n(n+1)(n+2)}{3}$;
5. $a_n = \frac{2n(n+1)(2n+1)}{3}$;
6. $a_n = \frac{n}{n+1}$.
7. $a_n = 3 - \frac{2n+3}{2^n}$.
8. $a_n = (n+1)! - 1$.
9. $a_n = 1 - \frac{1}{(n+1)!}$.

Solution 3. Create a descending string of equalities

$$
\begin{aligned}
&a_n = a_{n-1} + n^2,\\
&a_{n-1} = a_{n-2} + (n-1)^2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_3 = a_2 + 3^2,\\
&a_2 = a_1 + 2^2,\\
&a_1 = 1,
\end{aligned}
$$

and sum them up term-wise (the left-hand and right-hand sides separately). Having excluded the same terms from the both sides, we get the following formula for the sequence a_n:

$$a_n = 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 + n^2.$$

This is the direct formula of the sequence (a_n). However, it has a significant shortcoming: the amount of summands in the right-hand side increases when the number of elements in the left-hand side increases. It is desirable to find the reduced form of the above formula if it exists. For example, this can be achieved as follows. In the equality

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1,$$

we replace x by 1, 2, 3, ..., $n-1$, n in turn. We get the equalities:

$$
\begin{aligned}
&2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1,\\
&3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1,\\
&4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&n^3 - (n-1)^3 = 3 \cdot (n-1)^2 + 3 \cdot (n-1) + 1,\\
&(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n = 1.
\end{aligned}
$$

Summing them up term-wise we have:

$$(n+1)^3 - 1 = 3\cdot(1^2+2^2+\ldots+n^2)+3\cdot(1+2+\ldots+n)+n.$$

Taking into account that

$$1+2+3+\ldots+n = \frac{n(n+1)}{2},$$

we find a compact computational formula for $1^2+2^2+\ldots+n^2$:

$$1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution 6.

First Approach. Calculate several initial terms of the sequence. We have:

$$a_1 = \frac{1}{2};\ a_2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3};\ a_3 = \frac{2}{3}+\frac{1}{12} = \frac{3}{4};\ a_4 = \frac{3}{4}+\frac{1}{20} = \frac{4}{5}.$$

Having taken a glance at the sequence

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},$$

one easily recognizes the law that it follows: the numerators are the consequent natural numbers beginning with 1, and the denominators are greater than numerators by 1. This observation is the basis for the following hypothesis: the sequence is defined by the direct formula

$$a_n = \frac{n}{n+1}.$$

It remains to ensure that the recurrence relation

$$a_n = a_{n-1} + \frac{1}{n(n+1)}$$

holds for this formula.

We have:

$$\begin{aligned}&a_{n-1}+\tfrac{1}{n(n+1)} = \tfrac{n-1}{n}+\tfrac{1}{n(n+1)} = \tfrac{1}{n}\left(n-1+\tfrac{1}{n+1}\right) = \\ &= \tfrac{1}{n}\cdot\tfrac{n^2}{n+1} = \tfrac{n}{n+1} = a_n.\end{aligned}$$

Therefore, we conclude that our guess of the direct formula is correct.

Second Approach. According the given recurrence relation, we build a descending chain of equalities:

$$\begin{aligned}&a_n = a_{n-1}+\tfrac{1}{n(n+1)},\\ &a_{n-1} = a_{n-2}+\tfrac{1}{(n-1)n},\\ &a_{n-2} = a_{n-3}+\tfrac{1}{(n-2)(n-1)},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_3 = a_2+\tfrac{1}{3\cdot 4},\\ &a_2 = a_1+\tfrac{1}{2\cdot 3},\\ &a_1 = \tfrac{1}{1\cdot 2}.\end{aligned}$$

Summing them up term-wise, we get

$$a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}.$$

This sum can be reduced wittily provided by the equality

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

which holds for any k (except$k = 0$ and $k = -1$). Apply it to our sum:

$$a_n = \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \ldots + \left(\tfrac{1}{n-1}-\tfrac{1}{n}\right) + \left(\tfrac{1}{n}-\tfrac{1}{n+1}\right) =$$
$$= 1 - \tfrac{1}{n+1} = \tfrac{n}{n+1}.$$

Hint 7. Make sure that

$$a_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots + \frac{2n-1}{2^n}.$$

Then consider the equality

$$2a_n = 1 + \frac{3}{2} + \frac{5}{2^2} + \ldots + \frac{2n-1}{2^{n-1}}.$$

Subtract the previous equality from it term-wise.

Hint 8. Ensure that

$$a_n = 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \ldots + (n-1)\cdot(n-1)! + n\cdot n!.$$

Then make use of the equality

$$k\cdot k! = (k+1)! - k!,$$

which holds for any natural k.

Problem 1.142. *The sequence (a_n) is defined by the formula*

$$a_n = 1 - \frac{1}{(n+1)^2}.$$

Find the direct formula for the sequence (b_n), which is given by

$$b_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n.$$

Answer. $b_n = \frac{n+2}{2(n+1)}$.

Problem 1.143. *1. How many 10-digit numbers consist of 1 and 0 only?*

2. How many of them do not have two or more consecutive zeros?

3. How many 10-digit numbers from the first part of the problem do not have two or more consecutive ones?

Answer.

1. 2^9.

Hint. 2. Let γ_k be the amount of those k-digit numbers which consist of 0 and 1 only and do not have two or more consecutive zeros. Then

$$\gamma_1 = 1, \ \gamma_2 = 2$$

(there is one one-digit number of the defined type, which is 1, and two two-digit numbers: 10 and 11).

All sought k-digit numbers can be divided into two groups: those having the last digit 0 and those having the last digit 1. If a number has the last digit 0, then the penultimate digit is 1 (as two consecutive zeros are not allowed). The digit 1 can be preceded by any digit. Hence, the first $k-2$ digits of a number can form an arbitrary $(k-2)$-digit number (of the defined type). Thus, there are γ_{k-2} k-digit numbers in the first group. Now, let us find how many numbers are there in the second group, that is, how many numbers end with 1. As the digit 1 can be preceded by any of the two digits, there are γ_{k-1} such numbers. It appears that

$$\gamma_k = \gamma_{k-1} + \gamma_{k-2}.$$

This result solves the problem. The solution is almost complete. Really, basing on the observed recurrence relation accompanied by the initial conditions $\gamma_1 = 1, \ \gamma_2 = 2$, we can determine the numbers γ_k one by one:

$1, 2, 3, 5, 8, 13, 21, 34, 55, 98, \ldots$

In particular, $\gamma_{10} = 89$.

3. Denote the amount of such k-digit numbers by β_k. Then $\beta_1 = 1, \beta_2 = 1, \ \beta_k = \beta_{k-1} + \beta_{k+2}$. In particular, $\beta_{10} = 55$.

Problem 1.144. *How many 8-digit numbers do not have two or more consecutive zeros?*

Answer. 1242.

Hint. Let t_n denote the amount of such n-digit numbers. Following the solution of the previous problem with almost no changes, ensure that

$$t_n = t_{n-1} + t_{n-2}.$$

In addition, count one-digit and two-digit numbers having the required property. The result should be: $t_1 = 9$, $t_2 = 90$. Hence, (t_n) is defined by the same recurrence relation as the Fibonacci sequence but differs from it in initial conditions.

Problem 1.145. *How many 8-digit numbers do not have two or more consecutive even digits?*

Answer. 982.

Hint. Let τ_n be the amount of those n-digit numbers, which do not have two or more consecutive even digits. Ensure that $\tau_n = \tau_{n-1} + \tau_{n-2}$ and determine τ_1 and τ_2.

Problem 1.146. *How many 8-digit numbers do not have two or more consecutive odd digits?*

Answer. 917.

Problem 1.147. *Let $t(n)$ denote the number of points of intersection of n straight lines on the plane, among which there are no parallel lines and no three lines have a common point. Express $t(n)$ through $t(n-1)$. Determine $t(1)$. Having determined the recurrence relation defining the sequence $(t(n))$ and its initial term $t(1)$, calculate the elements of the sequence from $t(1)$ to $t(10)$. Basing on a recursive formula and the initial condition, find the direct formula of the sequence $(t(n))$.*

Answer. $t(1) = 0$, $t(n) = t(n-1) + (n-1)$; *the direct formula is* $t(n) = \frac{n(n-1)}{2}$.

Problem 1.148. *Let $s(n)$ denote the number of parts into which n straight lines split the plane, where there are no parallel lines and no three lines have a common point. Determine $s(1)$. Express $s(n)$ through $s(n-1)$. Having determined the recurrence relation defining the sequence $(s(n))$ and its initial term $s(1)$, calculate the elements of the sequence from $s(1)$ to $s(10)$. Basing on a recursive formula and the initial condition, find the direct formula of the sequence $s(n)$.*

Answer. $s(1) = 2$, $s(n) = s(n-1) + n$; *the direct formula is* $s(n) = \frac{n(n+1)}{2} + 1$.

Problem 1.149. *Circles are placed on the plane in such a way that:*

1. *all of them have the common point A;*
2. *any two of them have a common point in addition to the point A;*
3. *any three of them do not have common points except for A.*

   There are n circles. How many parts do these circles split the plane into? Find a recursive formula for the sought amount and basing on it determine the direct formula.

Answer. $1 + \frac{n(n+1)}{2}$.

Solution. Denote the wanted amount by t_n. We have $t_1 = 2$, $t_2 = 4$. Assume there are $k-1$ circles drawn. They split the plane into t_{k-1} parts. Observe how the amount of parts changes when the k-th circle is drawn. The new circle intersects with the previously drawn circles at k points: at the point A and at $k-1$ other points – one point belonging to each of those circles. These points split the new circle into k arcs, and each of these arcs splits the previously solid part of the plane into two parts. We conclude that if the circles are drawn one by one, then the k-th circle increases the amount of parts of the plane by k:

$$t_k = t_{k-1} + k.$$

Taking into account that $t_1 = 2$, we expand the discovered recurrence relation into the descending string of equalities:

$$\begin{aligned}
&t_n = t_{n-1} + n \\
&t_{n-1} = t_{n-2} + (n-1) \\
&\dots\dots\dots\dots\dots\dots \\
&t_3 = t_1 + 3 \\
&t_2 = t_1 + 2 \\
&t_1 = 2.
\end{aligned}$$

Summing them up term-wise we get the direct formula for t_n:

$$t_n = (n+(n-1)+\ldots+3+2+1)+1 = \frac{n(n+1)}{2}+1.$$

Problem 1.150. *Which is the maximum attainable number of parts that n circles can split the plane into? How the circles should be drawn to reach this?*

Answer. $2+n(n-1)$.

Solution. We can follow the algorithm presented in the solution of the previous problem. Assume there have been drawn $k-1$ circles, and they split the plane into τ_{n-1} parts. Which is the maximum number of parts which can be added by the next circle? Drawing this circle, we need to take care that it intersects each of the previous circles in two points and that these new points of intersection are different for different circles. In this case, these points of intersection will split the new circle into the maximum number of arcs, and each new arc will split the previously solid part of the plane into two parts. Therefore, we have

$$\tau_k = \tau_{k-1}+2(k-1).$$

Taking into account that $\tau_1 = 2$, we expand the obtained recurrence relation into descending string of equalities:

$$\begin{aligned}
&\tau_k = \tau_{k-1}+2(k-1),\\
&\tau_{k-1} = \tau_{k-2}+2(k-2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\tau_3 = \tau_2+2\cdot 2\\
&\tau_2 = \tau_1+2\cdot 1\\
&\tau_1 = 2.
\end{aligned}$$

Summing up these equalities term-wise we get the direct formula for τ_n.

Problem 1.151. *There are n circles on the plane and all of them have a common chord. Let σ_n be the amount of parts into which these circles split the plane. Find a recursive formula for σ_n. Apply it to determine the direct formula for σ_n.*

Answer. $\sigma_n = \sigma_{n-1}+2$, $\sigma_1 = 2$; $\sigma_n = 2n$.

Problem 1.152. *There are n circles inscribed at an acute angle on the plane. The first circle is arbitrary, the second passes through the center of the first, the third passes through the center of the second, etc. Finally, the last circle passes through the center of the penultimate one. Let δ_n denote the number of parts into which the circles split the angle. Find a recursive formula for δ_n. Basing on it, determine the direct formula for the sequence (δ_n).*

Answer. $\delta_n = \delta_{n-1}+4$, $\delta_1 = 3$; $\delta_n = 4n-1$.

Problem 1.153. *There are n circles inscribed in a strip (strip is a part of the plane bounded by two parallel lines. Every circle, except for the leftmost and the rightmost, passes through the centers of the adjacent circles. Let β_n denote the number of parts into which the circles split the strip. Derive a recursive formula for β_n. Apply it to determine the direct formula for β_n.*

Answer. $\beta_1 = 3$, $\beta_2 = \beta_1+4$, $\beta_n = \beta_{n-1}+5$ *for* $n \geq 3$; $\beta_1 = 3$, $\beta_n = 5n-3$ *for* $n \geq 2$.

Problem 1.154. *There is a path of length n dm and width 2 dm. The path is to be paved with 1 dm × 2 dm tiles. The tiles can be put widthwise or lengthwise. How many ways are there to pave the path (in terms of different patterns)? Derive a recursive formula for the sought number (denote it p_n) and specify the initial conditions.*

Answer. $p_1 = 2,\ p_2 = 2,\ p_n = p_{n-1} + p_{n-2}$.

Problem 1.155. *A path of length n dm and width 2 dm is to be paved with tiles. There are 10 dm × 30 dm and 20 dm × 30 dm tiles. One can use both types of tiles and put them widthwise or lengthwise. How many ways are there to pave the path? Denote the sought amount by q_n. Find a recursive formula for q_n and specify the initial conditions. Calculate eight initial terms of the sequence (q_n).*

Answer. $q_n = q_{n-1} + q_{n-2} + 3q_{n-3},\ q_1 = 1,\ q_2 = 2,\ q_3 = 6$; *1, 2, 6, 11, 23, 52, 108, 229.*

Problem 1.156. *1. There are n planes in the space which satisfy the following conditions:*

2. Any three planes have a unique common point.

3. Any four planes do not have common points.

a) How many lines are there, in which the planes intersect pairwise? Let a_n denote this number. Obtain a recursive formula for a_n. Basing on it, determine the direct formula for the sequence (a_n).

b) How many points of intersection of three planes are there? Denote this amount by b_n. Derive a recursive formula for b_n. Apply it to find the direct formula for the sequence (b_n).

c) How many parts do the planes split the space into? Denote the sought amount by c_n. Which recurrence relation defines the sequence (c_n)? Determine the direct formula for this sequence.

Answer.

1. $a_1 = 0,\ a_n = a_{n-1} + (n-1);\ a_n = \frac{n(n-1)}{2}$;
2. $b_1 = 0,\ b_n = b_{n-1} + \frac{(n-1)(n-2)}{2};\ b_n = \frac{n(n-1)(n-2)}{6}$;
3. $c_1 = 2,\ c_n = c_{n-1} + \frac{(n-1)n}{2} + 1;\ c_n = \frac{(n+1)(n^2-n+6)}{6}$.

Solution.

a) Assume there have been drawn $n-1$ planes and pairwise they have created a_{n-1} lines of intersection. We draw the n-th plane adhering to the stated rules. In particular, these rules provide that the new plane intersects with all others. Another requirement is that the new plane should not contain any of the available lines of intersection (otherwise, it would appear that three lines have a common line, hence, they have an infinite amount of common points). Therefore, the new plane adds $n-1$ new lines of intersection by crossing all available planes. This yields that

$$a_n = a_{n-1} + (n-1).$$

Replicating this equality for all indices smaller than n and taking into account that $a_1 = 0$, we get the following chain of equalities:

$$\begin{aligned}
&a_n = a_{n-1} + (n-1)\\
&a_{n-1} = a_{n-2} + (n-2)\\
&\dots\dots\dots\dots\dots\dots\\
&a_3 = a_2 + 2\\
&a_2 = a_1 + 1\\
&a_1 = 0.
\end{aligned}$$

Summing them up term-wise we receive that direct formula for the sequence (a_n).

b) Similarly to the previous part of the solution, we observe how the amount of points of intersection of three planes changes when the new plane is added.

So, let us have $n-1$ planes. According to the notation introduced in the statement of the problem, there are b_{n-1} such points. We add the n-th plane. It should not contain any of b_{n-1} points of intersection, but it should cross all the available lines of intersection. According to the previous result, there are $\frac{(n-1)(n-2)}{2}$ such lines. Hence, the new plane adds exactly the same amount of points of intersection and

$$b_n = b_{n-1} + \frac{(n-1)(n-2)}{2}.$$

Accompany the above recurrence relation with the initial condition $b_1 = 0$ to get the recursive definition of the sequence (b_n).

In order to get the direct formula for this sequence, we sum up the following equalities term-wise:

$$\begin{aligned}
&b_n = b_{n-1} + \tfrac{(n-1)(n-2)}{2}\\
&b_{n-1} = b_{n-2} + \tfrac{(n-2)(n-3)}{2}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&b_3 = b_2 + \tfrac{2\cdot 1}{2}\\
&b_2 = b_1 + \tfrac{1\cdot 0}{2}\\
&b_1 = 0.
\end{aligned}$$

We get:

$$\begin{aligned}
b_n &= \tfrac{(n-1)(n-2)}{2} + \tfrac{9n-2)(n-3)}{2} + \dots + \tfrac{2\cdot 1}{2} =\\
&= \tfrac{1}{2}[((n-2)^2 + (n-2)) + ((n-3)^2 + (n-3)) + \dots + (1^2 + 1)] =\\
&= \tfrac{1}{2}[(1^2 + 2^2 + \dots + (n-2)^2) + (1 + 2 + \dots + (n-2))] =\\
&= \tfrac{1}{2}\left[\tfrac{(n-2)(n-1)(2n-3)}{6} + \tfrac{(n-2)(n-1)}{2}\right] =\\
&= \tfrac{(n-2)(n-1)}{2\cdot 6} \cdot [(2n-3) + 3] = \tfrac{(n-2)9n-1)n}{6}.
\end{aligned}$$

c) Again, we follow the algorithm applied above. Let $n-1$ planes have already been placed, and the space is now split into c_{n-1} parts. The n-th plane will intersect all other

planes in the intersecting lines. These lines will intersect with each other in the maximum possible amount of points and will split the new plane into $\frac{n(n-1)}{2}+1$ parts (see Problem 1.148). Every part will split previously solid part of space into two parts. Hence

$$c_n = c_{n-1} + \frac{n(n-1)}{2} + 1.$$

Now, we form the familiar string of equalities:

$$\begin{aligned}
&c_n = c_{n-1} + \tfrac{n(n-1)}{2} + 1\\
&c_{n-1} = c_{n-2} + \tfrac{(n-1)(n-2)}{2} + 1\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&c_3 = c_2 + \tfrac{3\cdot 2}{2} + 1\\
&c_2 = c_1 + \tfrac{2\cdot 1}{2} + 1\\
&c_1 = 2,
\end{aligned}$$

ending with the initial condition. Summing up these equalities term-wise we obtain the direct formula for c_n:

$$c_n = n+1+\tfrac{2\cdot 1}{2}+\tfrac{3\cdot 2}{2}+\ldots+\tfrac{(n-1)(n-2)}{2}+\tfrac{n(n-1)}{2} =$$

$$= n+1+\tfrac{1}{2}\cdot(1^2+2^2+\ldots+(n-2)^2+(n-1)^2+1+2+\ldots+(n-1) =$$

$$= \tfrac{1}{6}(n+1)(n^2-n+6).$$

Problem 1.157. *Let n spheres have a common circle. How many parts do these spheres split the space into? Let f_n be the sought amount. Derive a recursive formula for f_n and appropriate initial terms. Find the direct formula for the sequence (f_n).*

<u>Answer.</u> $f_n = f_{n-1}+2,\ f_1 = 2;\ f_n = 2n.$

Problem 1.158. *There are two points on a sphere, through which n circles are drawn. All the circles belong to the sphere. How many parts do these circles split the sphere into? Let g_n be the sought amount. Derive a recursive formula for g_n and use it to determine the direct formula for the sequence (g_n).*

<u>Answer.</u> $g_n = g_{n-1}+2,\ g_1 = 2;\ g_n = 2n.$

Problem 1.159. *There are n spheres passing through the fixed points A and B. These points are the only common points for any three of these spheres. How many parts do the spheres split the space into? Let h_n be the sought number. Derive a recursive formula for h_n. Determine the direct formula for the sequence (h_n).*

<u>Answer.</u> $h_n = h_{n-1}+2(n-1),\ h_1 = 2;\ h_n = n^2-n+2.$

Problem 1.160. *There are n circles on a sphere. Any two of them intersect in two points and any three of them do not have a common point. How many parts do these circles split the sphere into? Find the direct and recursive formulas for the sought amount.*

<u>Answer.</u> $u_n = u_{n-1}+2(n-1), u_1 = 2;\ u_n = n^2-n+2.$

Problem 1.161. *Spheres are placed in the space in such a way that any two of them intersect (in a circle) and any three of them have only two common points. There are n spheres. How many parts do these spheres split the space into? Let v_n be the sought amount. Derive a recursive formula for v_n. Basing on it, find the direct formula for the sequence (v_n).*

Answer. $v_n = v_{n-1} + n^2 - 3n + 4,\ v_1 = 2;\ v_n = \frac{n(n^2-3n+8)}{3}$.

Chapter 2

Basic Concepts of Set Theory

Theory of sets laid the foundation of modern mathematics as a whole. We will not touch this substantive and profound theory in any depth. Instead, we will present a few of its fundamental concepts that will help us clearly and concisely formulate certain combinatorial results.

1. Sets

1.1. The Notion of a Set

A set is a distinct collection of certain things, creatures, symbols, or other objects. The objects that make up a set are called its elements. In order to make a distinction between one set and the others, one needs to know the pattern, which distinguishes the objects of this set from all other objects. It is in this sense that the words "distinct collection" are to be understood. Here are the examples of sets: the set of points of a given segment; set of vertices of a given triangle; set of all natural numbers; set of two-digit positive integers; set of letters of the Ukrainian alphabet; set of all the words used by Taras Shevchenko in the poem "The Caucasus", etc.

It is usual to denote sets by capital letters of various alphabets and their elements by lower case letters or other symbols. For the most important numeric sets there are fixed notation: N is used to denote the set of all natural numbers, Z is reserved for the set of all integer numbers, Q denotes the set of all rational numbers and R is a conventional notation for the set real numbers. In other cases, the notation is optional and should be clearly introduced before it is used.

If M is a set, then the formalized expression $a \in M$ (which reads "a is an element of M" or "a belongs to M") means that a is its element. The fact that a is not an element of the set M is expressed by the notation $a \notin M$. For example, the following expressions are correct: $2 \in N$, $2 \in R$, $\sqrt{2} \in R$, $\sqrt{2} \notin Q$, $\pi \notin Z$.

The equality sign can be placed between two symbols (letters) denoting sets only if they denote the same set. If A and B are sets and $A = B$, then it actually means that A and B is the same set. For instance, let A be the set of all two-digit natural numbers and B be the set of all natural numbers in the interval $(9, 100)$. Then $A = B$ as both sets have the same composition.

Depending on the number of their elements, sets can be finite or infinite. Finite sets are those, the number of elements of which can be expressed with a natural number. In other words, the set A is finite if it is possible to establish a bijection between its elements and the interval of the natural series (from 1 to some number n). For example, the set of one-digit natural numbers is finite. It is composed of 9 elements (numbers). The set of letters of Ukrainian alphabet is also finite. Other examples of finite sets include: integer solutions to the inequality $|x| \leq 10$; solutions of the equation $x^3 - 4x = 0$; possible dispositions of pieces on a chessboard, which can evolve during the game of chess. The latter set is incredibly large, but still it is finite. Nobody can count all possible combinations on the chessboard, even the most powerful computer. However, it is possible to find a number, which exceeds the number of such dispositions. We will make it below.

Infinite sets are those sets, for which it is impossible to establish a bijection between their elements and any interval of the natural series $[1, n]$. An attempt to count the elements of such a set inevitably turns into an infinite process. There are no infinite sets in the world around us. They are the creation of the human mind. The examples of infinite sets can be found only in mathematics. Here are some of them: N, Z, Q, R, the set of all points of any interval, the set of all diameters (line segments) of any circle, the set of all planes passing through a given point in the space, etc.

Given the subject of combinatorics, we will mostly be interested in finite sets. The number of elements of a finite set A is denoted by the symbol $|A|$. The difference between the symbols A and $|A|$ needs to be emphasized. A is a set, a collection of certain objects, while $|A|$ is the number of elements forming the set A. For example, if A is the set of all three-digit numbers, then $|A|$=900. By definition, the symbol $|C|$ makes no sense if C is an infinite set.

If a set is finite and contains a small number of elements, then the most natural way to define it is to list all its elements. In this case the elements of a set (or their names) are listed in arbitrary order and enclosed in braces. For example, the expression $\{2,3,5,7\}$ denotes the set of all one-digit prime numbers and the equality $P = \{1, 3, 5, 7, 9\}$ has the following meaning: the right-hand side informs that we are talking about the set of all one-digit odd numbers, and the left-hand side tells that the letter P denotes (or will denote further) this exact set.

Much more often than by direct listing, the set is defined by some characteristic property, which is intrinsic to all its elements and does not belong to any other object. The characteristic property is a distinctive feature distinguishing the elements of a set from all other objects. Under this approach to the definition of a set, it is denoted by:

$$\{x | W(x)\}.$$

The letter W denotes the characteristic (distinctive) property and the whole expression reads as follows: the set of those (elements) x, which have the property W. Obviously, the meaning of expression does not depend on the letter x. The set remains the same if we replace this letter with any other.

Example 2.1. *The expression $\{x | x^3 - x = 0\}$ denotes the set of roots of the equation $x^3 - x = 0$. Here, the equation serves as a characteristic property, which plays a crucial role in the definition of the set. An object (number) is an element of the set if and only if it is a*

solution to the equation. Having solved the equation, we can define the set by the list of its elements: $\{-1,1,0\}$.

Example 2.2. *The expression* $\{x \,|\, |x-5| < 1\}$ *denotes the set of those real numbers, for which the inequality* $|x-5| < 1$ *holds. Thus, it denotes the set of roots of this inequality. Its solutions form the open interval (without its ending points) of points, lying between the numbers 4 and 6. This set can also be denoted by (4, 6).*

Example 2.3. *The expression* $\{x \,|\, x \in Z\,, x < 0\}$ *denotes the set of all negative integer numbers.*

Does the symbol $\{x |\, W(x)\}$ always denote some set? For this to be true, at least W must be clearly defined, so that we can check its availability for any object. However, this is not a sufficient requirement, which is illustrated by the following example.

Let $A = \{x \,|\, x \in Z\,, x^2 = 2\}$.

Here, W is a property consisting of two parts: x is an integer, the square of which equals 2. There is no doubt that the property is clear and strictly defined. But there is no integer number, which possesses it, hence, A is not a set. Therefore, even with clear and acceptable property W the symbol $\{x |\, W(x)\}$ may not be a set and may mean nothing. This is a serious shortcoming. It would be much more convenient if this symbol always meant some set. Having such unification in mind, mathematicians introduced the notion of empty set. Instead of saying "no objects have the property W" we say "the set of objects having property W is empty". This is more than just another wording. Postulating the existence of the empty set we remove numerous exceptions and simplify operations with sets. We will ascertain this many times below. At this point, we underline that the empty set contains no elements and is unique. It is denoted by the symbol $\emptyset$. Using the introduced notion, we do not say that the set $\{x \,|\, x \in Z$ and $x^2 = 2\}$ does not exist, but that this set is empty: $\{x \,|\, x \in Z$ and $x^2 = 2\} = \emptyset$.

1.2. Subsets

If all elements of a set A belong to a set B, then A is called a subset of the set B. This is expressed as follows: $A \subset B$ (this expression reads "A is a subset of B"). It is also worth pointing out that for any set B the sets $\emptyset$ and B are subsets of the set B. This is provided by the definition. Really, any element of the set $\emptyset$ belongs to the set B, as there are no elements in $\emptyset$. Hence, $\emptyset \subset B$. Also, $B \subset B$. In this case, the condition in the definition turns into tautology. Other subsets of the set B (if any) are non-empty and do not coincide with the set B itself. They could be considered to be true parts of B. Such subsets are called proper non-empty subsets of the set B. For example, here is the list of all the subsets of the set $\{a, b, c\}$:

$$\emptyset, \quad \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

There are 6 proper non-empty subsets among them (all, except for the first and the last one).

Below, we provide several other examples of subsets.

Example 2.4. *Let R be the set of all real numbers. It has an infinite amount of subsets, including Q, Z, N, $[-1, 1]$ (the set of all numbers in the interval from –1 to 1 inclusive),*

$(0, \infty)$ *(the set of all positive numbers), the set of square roots of all natural numbers, the set of roots of all natural powers of 2, etc.*

Example 2.5. *Denote the set of all English words in the Oxford English Dictionary by C. Among its subsets there are: words, beginning with a consonant letter; two-syllable words; nouns; verbs; words, beginning with "a"; words with no closed syllables; words, having vowels at their start and end; words, having double consonant; nouns ending with the letter "o" and many others.*

Infinite sets have an infinite amount of subsets. This follows straightforwardly from the fact that they have an infinite amount of singletons, that is, subsets consisting of only one element. On the contrary, finite sets have a finite amount of subsets, which can be divided into groups by the number of their elements. Finding the amount of subsets in a certain group is one of the central combinatorial problems. We will solve it in the next section.

The notion of a set is twofold. From one point of view, a set is a collection of certain objects, which are called its elements. Alternatively, a set itself is an individual object, which in particular, can be an element of other sets. For instance, the group (set) of students of a given school consists of separate individuals and at the same time have patterns of the individual unit itself. Indeed, the groups of students from different schools can interact with each other similarly to separate individuals: organize sports or intellectual competitions, share the information and experience in self-regulation, free time activities, etc. The phrase "London school teams" sounds absolutely correct, despite formally it tells about the set, elements of which are other sets.

Let A be a set. Its subsets can be elements of other sets. In particular, it is possible to create a set consisting of all the subsets of the set A (and only of them). This set is called consistently: the set of all subsets of the set A. It is denoted by $S(A)$. For example, if $A = \{a, b, c\}$, then $S(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$. For any given set A, the sets A and $\emptyset$ are two of the elements of $S(A)$. In the case $A = \emptyset$, there is only one element in $S(A)$.

1.3. Intersection

Two sets A and B may have common elements. The set of such elements is called the intersection of the sets A and B and is denoted by the symbol $A \cap B$. So, by definition

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$$

(the set $A \cap B$ contains those elements of A that also belong to B). If A and B have no elements in common, then it is reasonable to assume that their intersection is empty set. This approach eliminates the exceptions. Under this agreement, the intersection of two sets is always a set (empty or non-empty).

Example 2.6. *The intersection of the set of all natural numbers divisible by 3 and the set of all natural numbers divisible by 5 is the set of natural numbers divisible by 15. In symbols, this assertion is given as follows:*

$$\{x|x \in N, x \vdots 3\} \cap \{x|x \in N, x \vdots 5\} = \{x|x \in N, x \vdots 15\}.$$

Example 2.7. *Let A and B be the sets of those students of a high school, who study English and German respectively. Then the set $A \cap B$ contains all those students, who study German and English at the same time. If there are no such students in the school, then the set $A \cap B$ is empty.*

Example 2.8. *Let A be the set of roots of the equation $(x+1)^2+(y-2)^2=5$, and B be the set of roots of the equation $2x+3y=-3$. The intersection $A \cap B$ consists of all the solutions to the system of equations*

$$\begin{cases} (x+1)^2+(y-2)^2=5, \\ 2x+3y=-3. \end{cases}$$

(Recall that the solution of an equation with two variables x and y is an ordered pair of numbers $(a;b)$, which turns the equation into correct numeric equality once x is replaced by a and y is replaced by b). To solve the system of equations is to find the elements of the set $A \cap B$.

Example 2.9. *If A is the set of solutions of the inequality $x^2-3x-10 \leq 0$, and B is the set of solutions of the inequality $x^2-6x-7 \leq 0$, then $A \cap B$ is the set of solutions of the system of inequalities*

$$\begin{cases} x^2-3x-10 \leq 0, \\ x^2-6x-7 \leq 0. \end{cases}$$

The operation of intersection "$\cap$" can be applied to more than two sets. In fact, we can "intersect" any amount of sets up to infinity. For example, let Z_6 the set of all integers divisible by 3, Z_{15} the set of all integers divisible by 15, and Z_{10} be the set of all integers divisible by 10. Then $Z_6 \cap Z_{15} \cap Z_{10}$ is the set of all integers divisible by 6, 10, 15 at the same time. A number is divisible by any of the above three numbers if and only if it is divisible by their least common multiple, which is 30. Hence,

$$Z_6 \cap Z_{15} \cap Z_{10} = Z_{30}.$$

1.4. Union

The elements of the sets A and B taken together, form the set which is called the union of the sets A and B and is denoted by $A \cup B$. That is

$$A \cup B = \{x | x \in A \text{ or } x \in B\}. \tag{2.1}$$

It is necessary to make two remarks here.

1. According to the definition of a set, every element of a set is unique. There can be no two identical elements in a set. In particular, if sets A and B have common elements ($A \cap B \neq \emptyset$), then each such element is presented in the union $A \cup B$ of these sets by only one element.

2. The conjunction "or" in English, as well as its correspondences in other languages, is used in two different contexts. "Would you like coffee with milk or black coffee", "It will be raining or dry", "You should wear boots or shoes to enter the restaurant", "We plant

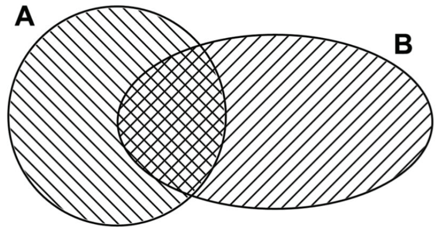

Figure 2.1. Intersection $A \cap B$, and the union $A \cup B$ of sets A and B.

rye or wheat in this area in autumn" – in all these phrases the conjunction "or" is used to suggest that only one possibility can be realized. In other words, in the above cases, the conjunction "or" combines alternative, incompatible options. However, sometimes this conjunction provides another, completely different meaning. "We hire for the positions of translators and assistants anyone who speaks German or French". There is no chance that someone will understand this announcement as if it only refers to those who speak only one of the two languages. The employer will gladly hire those who speak both languages. So, here the conjunction "or" is used in a different sense compared to the preceding examples. These two meanings can be called segregating and non-segregating respectively. The latter meaning is the one that is inherent in the definition of the union of sets A and B, expressed by formula (2.1). This definition should be understood as follows. An element x is included in $A \cup B$ in three cases, namely: when it belongs to A and not to B; when it belongs to B and not to A; and finally, when it belongs to both of these sets, that is, to their intersection. It is appropriate to illustrate the above by a schematic drawing (see Fig. 2.1).

Two circles depict the sets A and B. Their common part, looking like a biconvex lens, depicts an intersection $A \cap B$ (in the figure this part is crosshatched). The whole shaded figure made up of two crescents and a lens, corresponds to the union $A \cup B$. Schematic drawings used to illustrate the operations with sets are called the Euler diagrams (Eulerian circles) after the Swiss mathematician of the XVIII century, one of the most prominent creators of modern mathematics.

Below, we provide several examples illustrating the notion of union of sets.

Example 2.10.

$\{x | x \in Z \ and\, x \vdots 3\} \cup \{x | x \in Z \ and\, x \vdots 5\} = \{x | x \in Z \ and \ (x \vdots 3 \, or \, x \vdots 5)\}.$

Example 2.11.

$\{x | x \in R \ and\, x \geq 2\} \cup \{x | x \in R \ and\, x \leq -2\} = \{x | x \in R \ and \ |x| \geq 2\}.$

Example 2.12. *Let A be the set of those natural numbers that give a remainder of 1 upon division by* 3*, and B be the set of those natural numbers that give a remainder of* 2 *upon division by* 3. *Then* $A \cup B$ *be the set of those natural numbers that are not divisible by* 3.

Example 2.13. *Let l be a straight line on a plane* Π. *Let A be the set of straight lines on the plane* Π*, which are parallel to l (including l itself), and B be the set of straight lines on the plane* Π*, which are not parallel to l. Then* $A \cup B$ *is the set of all straight lines on* Π.

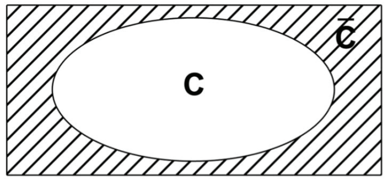

Figure 2.2. Complement of the set C.

The operation of union can be applied to three, four, or any greater amount sets. In these cases, the definition does not change significantly.

1.5. Difference

The difference of A and B (also called the set-theoretic difference of A and B, or relative complement of B with respect to A), denoted by $A\backslash B$, is the set of all elements that are members of A but are not members of B. Thus,

$$A\backslash B = \{x|x \in A\ , x \notin B\}.$$

In order to get the set $A\backslash B$, one needs to remove from the set A all those elements, which belong to B. On the Euler diagram Fig. 2.1 the set $A\backslash B$ is a shaded crescent. If $A \cap B = \emptyset$, then naturally $A\backslash B = A$.

Example 2.14. *Let A be the set of all straight lines in space, let l be one of these lines, and let B be the set of those lines in space, which lay in the same plane with l. In other words, a line t belongs to B if and only if there exists a plane, containing t and l. In this case, $A\backslash B$ consists of lines, which are skew to the line l, and $B\backslash A$ is an empty set.*

Example 2.15. *If N_3 is the set of those natural numbers, which are divisible by 3, N_2 is the set of even natural numbers, then $N_3\backslash N_2$ is the set of all those odd natural numbers, which are divisible by 3, and $N_2\backslash N_3$ is the set of those even natural numbers, which are not divisible by 3.*

Example 2.16. *Let A be the set of those three-digit natural numbers, which have at least one even digit, and B be the set of those three-digit natural numbers, which have at least one odd digit. Then $A\backslash B$ is the set of all those three-digit numbers, all digits of which are even, and $B\backslash A$ is the set of all those three-digit numbers, all digits of which are odd.*

Example 2.17. *The difference $R\backslash Q$ is the set of irrational numbers.*

1.6. Complement

Within the framework of various sciences or problems, instead of investigation of arbitrary sets, one has to deal with subsets of a given "superset". For example, for arithmetic or analysis of functions of one argument, such a superset is the set of all real numbers R, and

for planimetric this is the set of all points of the plane (or the set of all pairs of real numbers denoted by $R \times R$).

Let E be a superset (it is also called a universal set). We emphasize that despite its pretentious name this is not a special set. Declaring E as a superset, we simply commit ourselves to consider only those sets that are subsets of E. Such sets form the set of all subsets of the set E, denoted by $S(E)$. Subsets of E are elements of $S(E)$, and vice versa. The subsets of the set E have the following property: the union, intersection, and difference of any two of them are again subsets of E. In symbols this property is expressed as follows: if $A \in S(E)$ and $B \in S(E)$, then $A \cap B \in S(E)$, $A \cup B \in S(E)$ and $A \backslash B \in S(E)$. Applying operations "$\cap$", "$\cup$" and "$\backslash$" to the elements of $S(E)$ (with the subsets of the set E) we always get elements of this very set (the set of all subsets of E).

Among various differences $A \backslash B$ of sets from $S(E)$ those with the minuend being the set E itself are of special importance. The difference $E \backslash C$ is called the complement of the set C (to the whole superset E). It is denoted by $\overline{C}$. In Fig. 2.2 the set E is depicted by a rectangle, and the set C is depicted by a circle. The complement $\overline{C}$ of the set C is shaded on the figure. According to the definition, the sets C and $\overline{C}$ do not intersect (that is $C \cap \overline{C} = \emptyset$) and together they form the whole superset E ($C \cup \overline{C} = E$).

Example 2.18. *Let R be a superset. Then:*

a) $\overline{Q}$ is the set of all rational numbers.

b) If $A = \{x | x < 1\}$, then $\overline{A} = \{x | x \geq 1\}$.

c) $\overline{Z}$ is the set of all non-integer numbers.

Example 2.19. *Let the universal set (superset) be the set of all straight lines in space.*

a) Let A be the set of those straight lines, which intersect with the given line l. Then $\bar{A}$ is the set of all lines, skew to l or parallel to it (including the line l itself).

b) If C is the set of all straight lines in space, which are parallel to the given plane τ or lay on this plane, then $\bar{C}$ is the set of those lines, which intersect with the plane τ in one point.

c) If D is the set of those lines in space which form a sharp angle with the given line l, then $\bar{D}$ consists of the lines, which are parallel or orthogonal to it.

1.7. Cartesian Product

A Cartesian product of sets A and B (order is essential) is the set of all ordered pairs $(a; b)$, where the first component belongs to A, and the second belongs to B. This set is denoted by the symbol $A \times B$. Thus,

$$A \times B = \{(x; y) \,|x \in A,\ y \in B\}.$$

For example, if $A = \{a, b, c\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a; 1),\ (a; 2),\ (b; 1),\ (b; 2),\ (c; 1),\ (c; 2)\}.$$

If a set A contains m elements, and a set B contains n elements, then according to the combinatorial rule of product the set $A \times B$ contains $m \cdot n$ elements.

Two-dimensional (plane) objects, in particular, the rectangles, split by longitudinal and transverse lines on smaller squares (cells), provide a good illustration of the Cartesian product of two sets. Recall the chessboard (see Fig. 2.3).

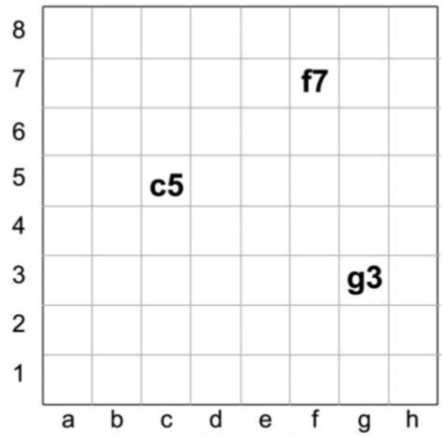

Figure 2.3. Chessboard.

There is a square board, split by vertical and horizontal lines into 8 rows ("ranks" in chess terminology) and 8 columns ("files"). The files are denoted by eight initial letters of the English alphabet from left to right: a, b, c, d, e, f, g, h. Similarly, numbers are used to denote ranks:$\{1, 2, 3, 4, 5, 6, 7, 8\}$ from bottom to top. Thus, each cell (the chessboard square) has its name consisting of two components: a letter and a number. The rule for naming cells is the same as for points in the coordinate plane when they are assigned two coordinates (abscissa and ordinate) each. The name of the cell is composed of the names of file and rank intersecting at it. Thus the names of cells are the elements of the Cartesian product of the sets $A = \{a, b, c, d, e, f, g, h\}$ and $N_8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. In addition, naming the cells of a chessboard we ye all the elements of the set $A \times N_8$. Thus, the rule for naming establishes a bijection between the set of squares of a chessboard and the elements of the set $A \times N_8$.

Similarly, the rule for assigning coordinates to the points on the coordinate plane establishes a bijection between the set of the points on the plane and the elements of the Cartesian product $R \times R$ (the pairs of real numbers).

The rule of creation of the Cartesian product of two sets is easily generalized for the case of more sets (Cartesian factors). In particular, the Cartesian product of three sets A, B and C is the set containing all the triplets $(x; y; z)$, where $x \in A$, $y \in B$ and $z \in C$. Naturally, this set is denoted by $A \times B \times C$. Thus,

$$A \times B \times C = \{(x; y; z) \,|x \in A,\ y \in B,\ z \in C\}.$$

For example, $R \times R \times R$ (or briefly R^3) is the set of triplets $(x; y; z)$ of real numbers. Its "geometric twin" is the set of all points of the coordinate space.

2. Correspondence

Let A and B be sets. There is a correspondence between the sets A and B if some (or all) elements of the set A is connected to some elements of the set B by a certain rule. The

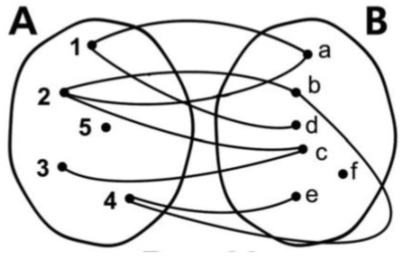

Figure 2.4. Correspondence.

diagram in Fig. 2.4 schematically illustrates the notion of correspondence. The diagram reads that element 1 of A is related to the elements a and d of the set B, and element 2 of the set A is related to the elements a, b and c of B. Similarly, we can see that the element c from B is related to elements 2 and 3 from A, and the element 5 from A is not related to any element from B.

Overall, Fig. 2.4 provides complete information about one of many possible correspondences between the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f\}$. This schematic illustration of correspondence is called a mapping diagram. On the mapping diagram, related elements of two different sets are connected by lines, while no lines are connecting the unrelated elements. This vivid illustration of correspondence demonstrates, which means are there to define it except the explicit presentation. Instead of connecting related elements with lines, we can simply name the pairs of matching elements. Here is the set of pairs (an element of A; an element of B), which describes the correspondence between A and B, depicted above by the graph in Fig. 2.4:

$$\{(1; a),\ (1; d),\ (2; a),\ (2; b), (2; c),\ (3; c),\ (4; b),\ (4; e)\}.$$

We arrive at a unified and inherently set-theoretical method of definition of one or another correspondence between arbitrary sets A and B. Any such correspondence can be defined by an appropriate set of pairs $(x; y)$, where the first component belongs to A and the second one belongs to B. Conversely, any such set of pairs defines a certain correspondence between A and B. But any such set of pairs is a subset of the set of all pairs, which is the set $A \times B$. Therefore, we arrive at the following, to an extent, elegant conclusion:

Any correspondence between the sets A and B is defined by a subset of the Cartesian product $A \times B$. There is a bijection between correspondences between sets A and B and the subsets of the set $A \times B$.

Let the set A contain m elements, and the set B contains n. We draw m vertical and n horizontal lines. Assume that the former are the elements of the set A and the latter are the elements of B. Then the points of intersection of these lines are defined by pairs $(x; y)$, the first component of which belongs to A, and the second belongs to B. The set of all these points is the explicit illustration of the Cartesian product $A \times B$. This illustration can be called a netting image of $A \times B$. If we take some correspondence between the sets A and B, then there is a subset of the Cartesian product $A \times B$ related to it, and this subset corresponds in turn to a subset of points of the netting image of $A \times B$. This set is called

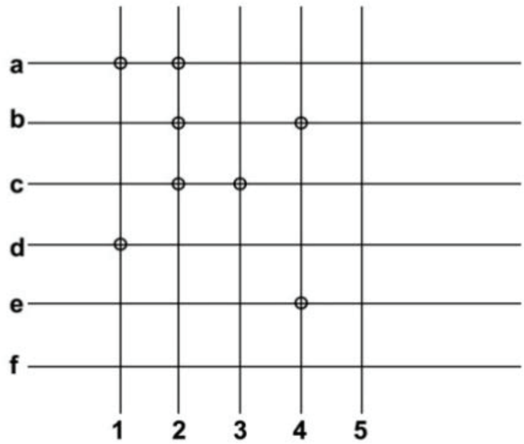

Figure 2.5. Graph of correspondence.

the graph of correspondence. In Fig. 2.5 there is the graph of a correspondence between the sets $A = (1,2,3,4,5)$ and $B = (a,b,c,d,e,f)$, which we have considered above (its mapping diagram is in Fig. 2.4). Conventionally, a correspondence between two sets is denoted by some letter or other symbol. If correspondence between sets A and B is denoted by φ, then it is expressed as $A\varphi B$. For example, let φ be the correspondence between the sets A and B given by the graph in Fig. 2.5.

Then $1\varphi a$ means that the elements 1 and a correspond to each other (match each other, are paired with each other). When the symbol $\bar{\varphi}$ is put between some elements from A and B, this means that these elements are not related to each other. For instance, $1\bar{\varphi}f$, $3\bar{\varphi}a$ etc.

Example 2.20. *Examples of correspondences.*

Let A be the set of all straight lines in space and let B be the set of all planes.

1. *Let $A\pi B$ be the correspondence between these sets, under which a line $l \in A$ and a plane $\pi \in B$ are matched if they are parallel to each other (including the case when l belongs to the plane π). Under this correspondence, any line has infinite amount of planes related to it. And vice versa: any plane has infinite amount of lines paired with it.*

2. *Another important from the geometric point of view correspondence is the one, which relates a line to the planes, which it crosses. This time any line again has infinite amount of matching planes, and any plane has infinitely many lines paired with it.*

3. *Let M be a given point in space. We claim that related to a line l are those planes, which possess both following properties: they are orthogonal to l and pass through the point M. Under such correspondence between lines and planes, there is only one plane matched with any line. However, there are infinitely many lines corresponding to any plane passing through the point M. Other planes are not matched with any lines.*

4. *Now, we relate every plane to the lines, which are orthogonal to it and contain the point M. This time we get the correspondence between A and B, under which there is only one line related to any given plane. On the other side, every line l either has infinitely many planes corresponding to it, or has none, depending on the point M laying outside this line or on it.*

5. *Let M and N be points in space. We match any line passing through the point M with the plane, which is orthogonal to this line and passes through the point N. We get a certain correspondence between lines and planes in space. Given this correspondence, every line passing through the point M is paired with one plane. Any other line is not paired with any plane. The same applies to planes. Those of them, which contain the point N, are related with one line each. Alternatively, if the plane does not pass through the point N, then there are no lines corresponding to it. Such correspondence between the sets A and B establishes a bijection of the lines containing the point M, and the planes passing through the point N.*

Example 2.21. *Examples of correspondences.*

Let A be the set of letters of the English alphabet, and B be the set of English words in the Oxford English Dictionary. Below we present several "natural" correspondences between A and B.

1. *Every letter is matched with all the words containing it, and every word is related to all the letters forming it.*

2. *Every letter is related to all the words where it doubles. In turn, all the words are matched to letters, which double in them. According to this correspondence, most words are "isolated" (that is, they do not have matches in the set of letters). Because the words having doubling letters, form quite a minor part of all the words. Similarly, some of the letters do not have corresponding words (e.g., "j", "y").*

3. *Every letter corresponds to the words beginning with it, and every word is paired with its initial letter*

4. *Every letter is matched with the words, in which it appears in an open syllable (and every word is matched with the letters in its open syllables). The words "fan" and "salt" do not have matches. The word "hero" has the letters "e" and "o" corresponding to it.*

   In the context of the notion of correspondence between sets A and B, these sets can be the same. In this case, we can deal with correspondence between the elements of the set A. Every correspondence of this type is completely defined by some subset of the Cartesian product $A \times A$.

Example 2.22. *Example of correspondence on the set A.*

Let A be the set of English words in the Oxford English Dictionary. We provide some examples of correspondence between the elements of this set below.

1. *Every word is matched with all words sharing at least one letter with it.*

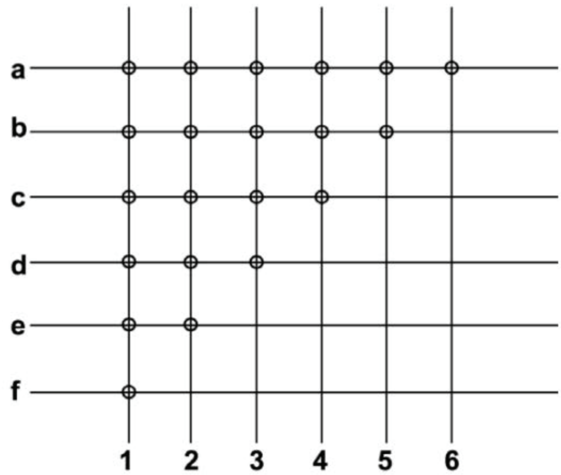

Figure 2.6. Ascending order. Graph of correspondence.

2. *Every word corresponds to all words beginning with the same letter as it is. This is so-called classifying correspondence, which divides all the words into groups (classes) following the alphabetical principle.*

3. *Every word is matched with all words of the same part of speech. This is another example of classifying correspondence, as all the words are divided into classes by the part of speech, which they represent (nouns, verbs, adjectives etc.).*

4. *Every word corresponds to all words, which differs from it in one letter.*

5. *Every word is related to the words, following it in alphabetical order. Besides, this example illustrates that correspondence between the elements of A can be adequately defined with a diagram if and only if the lines connecting certain elements are supplemented by appropriately placed arrows (in one or maybe two directions).*

Example 2.23. *Example of correspondence on the set A.*
We proceed with several examples of correspondences on the set $Z_6 = \{1, 2, 3, 4, 5, 6\}$.

1. *Ascending order: every number is matched with all smaller numbers. The graph of such correspondence is presented in Fig. 2.6.*

   It is an explicit illustration of the following subset of the Cartesian product $Z_6 \times Z_6$*:* $\{$*(1; 1), (1; 2), (1; 3), (1; 4), (1; 5), (1; 6), (2; 2) (2; 3), (2; 4), (2; 5), (2; 6), (3; 3), (3; 4), (3; 5), (3; 6), (4; 4), (4; 5), (4; 6), (5; 5), (5; 6), (6; 6)*$\}$. *This set defines the ascending order.*

2. *Divisibility: every number corresponds to its divisors. The mapping diagram and the graph of this correspondence is shown in Fig. 2.7.*

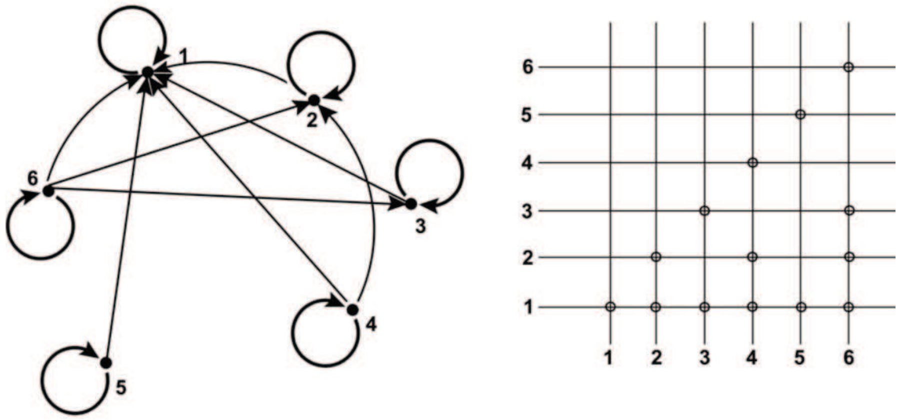

Figure 2.7. Divisibility: every number corresponds to its divisors. Graph of correspondence.

3. *Every number is matched with the numbers, which differ from it by 1 at most. Here is the subset of the Cartesian product $Z_6 \times Z_6$, which defines this correspondence:*$\{$*(1; 1), (1; 2), (2; 1), (2; 2), (2; 3), (3; 2), (3; 3), (3; 4), (4; 3), (4; 4), (4; 5), (5; 4), (5; 5), (5; 6), (6; 5), (6; 6)*$\}$*. The mapping diagram and the graph of this correspondence is shown in Fig. 2.8.*

4. *"+1" correspondence: every number is paired with the number exceeding it by 1. This correspondence defines the following subset of the Cartesian product $Z_6 \times Z_6$:* $\{(1;2),(2;3),(3;4),(4;5),(5;6)\}$. *Draw the mapping diagram and the graph of this correspondence.*

2.1. Mapping

Let A and B be sets, which may coincide. A correspondence φ is called a mapping of the set A to the set B if this correspondence provides that every element of A is matched to exactly one element of B. One can develop the correct understanding of the notion of mapping by carefully reading through the above definition and comparing it with the diagram shown in Fig. 2.9.

The characteristic (defining) property of the diagram of a mapping of A to B, which distinguishes it from the diagrams of other correspondences, is that every point of the set A is connected with one (exactly) line with some point from the set B. The definition of mapping is asymmetrical in respect of the sets A and B. It applies certain restrictions on the behavior of the elements of A, and in no way restricts the behavior of the elements of B. Every element of A should necessarily have only one matching element from B, while the elements of the set B may behave rather freely: there can be such elements, to which there are no matches in A at all, or alternatively, there can be elements with multiple matches. This asymmetry is preserved by the way in which mapping is denoted:

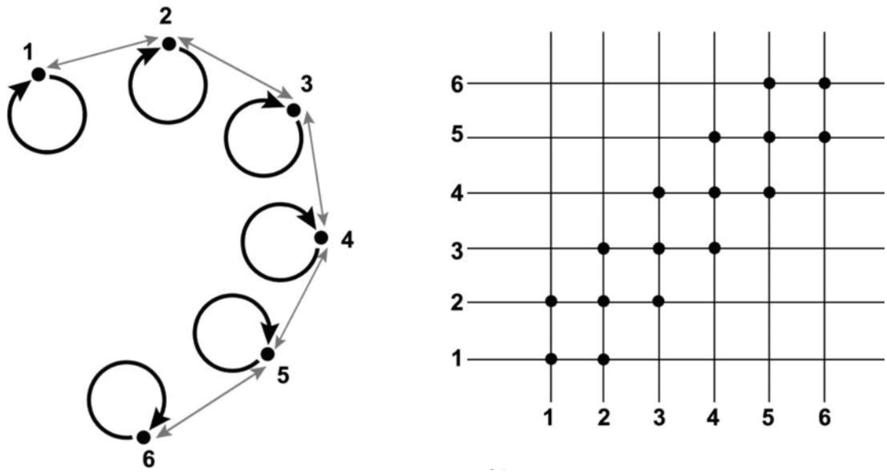

Figure 2.8. Every number is matched with numbers, which differ from it by 1 at most. Graph of correspondence.

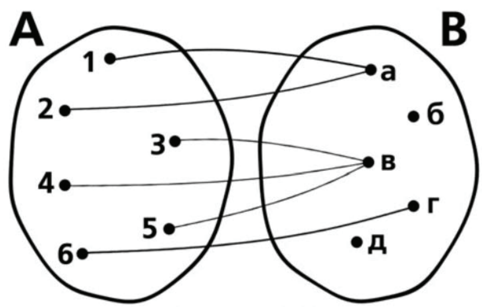

Figure 2.9. Mapping.

$$\varphi : A \to B$$

(this reads: the mapping φ of (the set) A to (the set) B). If $a \in A$, then the element of B corresponding to it is denoted by $\varphi(a)$ and is called the image of (the element) a under the mapping φ. Let P be a subset of A. The set of images of all elements from P is denoted by $\varphi(P)$ and is called the image of the set P. In particular, $\varphi(A)$ is the image of the whole set A. If $b \in B$ then $\varphi^{-1}(b)$ denotes the set of those elements of A, the image of which under the mapping φ is b. The set $\varphi^{-1}(b)$ is called the preimage (or inverse image) of the element $b \in B$ under the mapping $\varphi : A \to B$. The elements of the set $\varphi^{-1}(b)$ are also called the preimages (inverse images) of the element b, provided that this set is non-empty.

In the case of the mapping $\varphi : A \to B$, the diagram of which is shown in Fig. 2.9, we have:

$$\varphi(1) = \varphi(2) = a,\ \varphi(3) = \varphi(4) = \varphi(5) = c,\ \varphi(6) = d;$$
$$\varphi(\{2,4\}) = \{a,b\},\ \varphi(\{4,5\}) = \{c\},\ \varphi(A) = \{a,c,d\};$$
$$\varphi^{-1}(a) = \{1,2\},\ \varphi^{-1}(b) = \varphi^{-1}(e) = \emptyset,\ (c) = \{3,4,5\},$$
$$\varphi^{-1}(d) = \{6\}.$$

A mapping $\varphi : A \to B$ is called injective if any two different elements from A have different images. An injective mapping is called an injection. The same property can be put differently: a mapping $\varphi : A \to B$ is an injection if the preimage of every element from B is either empty or one-element set.

A mapping $\varphi : A \to B$ is surjective (or is a surjection) if $\varphi(A) = B$. In other words: a mapping $\varphi : A \to B$ is a surjection if there are no elements in B, the preimage of which is an empty set.

A mapping $\varphi : A \to B$, which is surjective and injective is called a bijection. This type of correspondence between two sets was covered in detail in one of the previous chapters.

Example 2.24. *Below we present several examples of mappings of the set of all English words in the Oxford English Dictionary to the set of all natural numbers N.*

1. *Every word is matched with its serial number in the Dictionary. This mapping is injective.*
2. *Every word is matched with the number of letters creating it (e.g., φ(rectangle)=9, φ(circle)=6 etc.).*
3. *Every word is matched with the number of different letters creating it (e.g., φ(rectangle)=8, φ(circle)=5 etc.).*
4. *Every word is matched with the number of page, on which it appears in the first edition of the Dictionary.*

Example 2.25. *The mappings of the set Z to Z are the following:*

1. *Every integer is matched with its absolute value. The image of the set Z under this mapping is the set of all natural numbers N along with 0. The mapping is not injective (the numbers* 1 *and* -1 *have the same absolute value) and is not surjective (there is no preimage for* -1).
2. *Every integer x is paired with the number* $x+1$. *This mapping is a bijection.*
3. *Every integer number x is paired with the number* $2x$ *the image of Z is the set of all integer even numbers. The mapping is injective but not surjective.*
4. *Every even number is matched with half of it and every odd number is matched to itself. The mapping is surjective but not injective.*

Problems

Problem 2.1. *Name all elements of the set of all two-digit prime numbers.*

Problem 2.2. *List the elements of the sets defined with the characteristic properties of their elements.*

1. $\{x|x \in Z \text{ and } |x| \leq 4\}$;
2. $\{x|x \in N \text{ and } |x-4| \leq 5\}$;
3. $\{x|x \in N \text{ and } x^2 < 50\}$;
4. $\{x|x \in Z \text{ and } 3 \leq |x-7| \leq 6\}$;
5. $\{x\,|3x^3 - 5x^2 - 2x = 0\}$;
6. $\{x\,|6x^4 - 13x^2 + 6 = 0\}$;
7. $\{x\,|x^5 - 5x^3 + 6x = 0\}$;
8. $\{x|x \in Z \text{ and } x^2 + 5x - 14 < 0\}$.

Answer.

1. $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$;
2. $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
3. $\{1, 2, 3, 4, 5, 6, 7\}$;
4. $\{1, 2, 3, 4, 10, 11, 12, 13\}$;
5. $\{-\frac{1}{3}, 0, 2\}$;
6. $\left\{\sqrt{\frac{3}{2}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -\sqrt{\frac{3}{2}}\right\}$;
7. $\{0, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}\}$;
8. $\{$-6, -5, -4, -3, -2, -1, 0, 1$\}$.

Problem 2.3. *List all subsets of the set* $\{a, b, c, d\}$.

Answer. *There are 16 subsets including the empty set and the whole set. The sets are:* $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ *and* $\{a, b, c, d\}$.

Problem 2.4. *Let A be a set,* $a \in A$*. Denote the set of subsets of A, which does not contain the element a by L, and the set of subsets of A containing a by M. Establish a bijection between the sets L and M.*

Solution. We match each subset T of the set A, which does not contain a, with the set $T \cup \{a\}$ which includes it. This pairwise matching of the elements of L with the elements of M is a bijection between L and M. Indeed, we pair those otherwise similar subsets of A from L and M, which differ only by one element, – the element a. One subset of a pair has it, while the other does not. Removing the element a from all the subsets containing it, we are getting all the subsets, which do not contain it. Conversely, adding this element to all the subsets missing it, we are getting all the subsets, which include it.

Problem 2.5. *A set A is composed of n elements. How many subsets are there in it?*

Solution. To find the answer, we will use the above result (see Exercise 4) about two types of subsets: those that contain a certain element and those that do not. Actually, the previous problem provides a key to the answer to this one. Basing on it, it is quite straightforward to derive a recurrence relation for the sought number. Really, let $p(n)$ be the sought number, which is the amount of subsets of a set of n elements. We choose a to be any element of our set and ask ourselves: how many subsets of the set A does not contain the element a? Such subsets compose the set $S(A \backslash \{a\})$. In other words, they form the set of all subsets of the set $A \backslash \{a\}$. The latter contains $n-1$ elements, hence using our notation, it has $p(n-1)$ subsets. Therefore, there is the same amount of those subsets of the set A, which does not include a. But the solution to the previous problem revealed that the amount of subsets of the set A containing a is the same. Therefore,

$$p(n) = 2p(n-1).$$

Thus, we have the one-step formula for $p(n)$. It remains to determine the initial condition. For $n = 1$, the set A has to subsets: $\emptyset$ and A. So

$$p(1) = 2.$$

Multiplying term-wise the equalities of the following descending chain

$$\begin{aligned}
&p(n) = 2p(n-1)\\
&p(n-1) = 2p(n-2)\\
&p(n-2) = 2p(n-3)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&p(3) = 2p(2)\\
&p(2) = 2p(1)\\
&p(1) = 2,
\end{aligned}$$

we get the direct formula for the amount of subsets:

$$p(n) = 2^n.$$

This is an outstanding result, because it reveals the fundamental property of finite sets: the dependence of number of subsets on the number of elements.

There is another way to derive this formula.

Solution II. Line up the elements of the set A and enumerate them from left to right. We get the following string of elements:

$$a_1\ a_2\ a_3\ a_4\ \dots\ a_{n-1}\ a_n.$$

Now, we want to create a subset out of these elements. What our course of action should be? Obviously, we have to define the elements composing it. For example, we can proceed as follows: we place the "+" sign under the element a_i of the above string if we include it in the new set, and "–" otherwise. A sequence of length n (according to the amount of elements a_i) consisting of "+" and "–g" signs will define a subset for us. Conversely, any subset of the set A have a sequence of signs "+" and "–" of length n corresponding to it. Thus, there is a bijection between the subsets of the set A and these sequences, which evidences that there are the same amounts of objects of both types. Having counted the amount of sequences, we know the number of subsets.

The problem about the sequences of "+" and "–" signs can be formulated as follows. Let us have a string of cells (a rectangle split into cells (squares)). There are n cells. How many ways are there to place "+" and "–" signs into the cells (one sign per cell)? This situation appears to be familiar. This problem seems to be specifically designed for the rule of product application. There are two options for each position (cell) – "+" or "–" sign. Moreover, any of the signs can appear in any position independently from the choices made for other positions. According to the rule of product, "+" and "–" signs can be filled in the cells of the string in 2^n ways. Therefore, an n-element set has the same amount of subsets.

Problem 2.6. *A set A consists of n elements* $(n > 2)$. *There are elements a and b among them. How many subsets of the set A are there, which:*

1. *do not include any of the elements a and b?*
2. *contain exactly one of the elements a and b?*
3. *contain both elements a and b?*

Answer.

1. 2^{n-2};
2. 2^{n-1};
3. 2^{n-2}.

Problem 2.7. *A set A contains n elements, and its subset B has k elements. How many subsets of the set A are there:*

1. *which do not intersect with B?*
2. *which include B?*
3. *have common elements with B but do not include the whole set B?*

Answer.

1. 2^{n-k};
2. 2^{n-k};

3. $2^{n-k+1}(2^{k-1}-1)$.

Problem 2.8. *A set A contains n elements, and its subsets B and C contain k and s elements respectively. In addition, the sets B and C do not have common elements* $(B \cap C = \emptyset\}$. *How many subsets of the set A:*

1. *do not intersect with B and C?*
2. *do not intersect with B?*
3. *include both sets B and C?*
4. *include the set C?*

Answer.

1. 2^{n-k-s};
2. 2^{n-k};
3. 2^{n-k-s};
4. 2^{n-s}.

Problem 2.9. *Which are the necessary and sufficient conditions for the following equalities to hold:*

1. $A \cap B = A$;
2. $A \cup B = A$;
3. $A \backslash B = A$?

Answer.

1. $A \subset B$;
2. $B \subset A$;
3. $A \cap B = \emptyset$.

Problem 2.10. *Construct the Euler diagrams to verify the following equalities:*

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 2.11. *Does the following equality hold for arbitrary sets*

$$A \backslash B = A \backslash (A \cap B)?$$

Problem 2.12. *Find the set*

$$(A \backslash B) \cup (B \backslash A)$$

on the Euler diagram.

Problem 2.13. *Let A and B be sets of m and n elements respectively* $(|A| = m,\ |B| = n)$.

1. *Find the condition for the equality* $|A \backslash B| = m - n$ *to hold?*
2. *Which values can the variable* $|A \backslash B|$ *attain depending on the mutual positioning of the sets A and B?*

Answer.

1. $B \subset A$;
2. *The given variable can be integer number from* $[0, m]$.

Problem 2.14. *A and B are sets, having m and n elements respectively. How can the value of the variable* $|A \cap B|$ *vary depending on the mutual positioning of the sets A and B?*

Answer. $|A \cap B|$ *can attain integer values from the interval* $[0, \min\{m, n\}]$. $|A \cap B| = 0$, *when* $A \cap B = \emptyset$; $|A \cap B| = \min\{m, n\}$, *when of the sets A and B is a subset of the other.*

Problem 2.15. *Construct the Euler diagrams to ensure that the following equalities hold:*

1. $A \backslash (B \cup C) = (A \backslash B) \backslash C$;
2. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

Problem 2.16. *Which are the necessary and sufficient conditions for the following equalities to hold:*

1. $A \backslash B = \emptyset$;
2. $A \cup B = \emptyset$;
3. $A \backslash B = B$.

Answer.

1. $A \subset B$;
2. $A = B = \emptyset$;
3. $A = B = \emptyset$.

Problem 2.17. *Let A and B be finite sets and* $|A| = m$, $|B| = n$. *Which values can the variable* $|A \cup B|$ *attain depending on the mutual positioning of the sets A and B?*

Answer. *The set* $A \cup B$ *can have any amount of elements from* $\max\{m, n\}$ *to* $m + n$.

Problem 2.18. *State the condition for the equality* $|A \cup B| = |A| + |B|$ *to hold (A and B are finite sets).*

Answer. $A \cap B = \emptyset$ *(A and B do not have common elements).*

Problem 2.19. *Prove that for any finite sets A and B the following equality holds (the inclusion-exclusion principle)*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof. We have to prove that on the right-hand side of the equality each element of the union $A \cup B$ is accounted for once. All elements of the union can be split into three classes: the first includes those elements, which belong to A and do not belong to B; the second incorporates those elements of B, which do not belong to A; finally, the third class comprises the elements, which belong to both sets A and B, and thus are the elements of the intersection $A \cap B$. Let a be an element of the first class. It is accounted for once in the amount $|A|$ and is not accounted for in $|B|$ and $|A \cap B|$. Therefore, on the right-hand side of the hypothetical equality, this element is accounted for $1+0-0=1$ times. We have the same situation with any element of the second class. Now, let c is an element of the third class. It is accounted for once in all of the amounts $|A|$, $|B|$ and $|A \cap B|$, hence there is $1+1-1=1$ instance of it on the right-hand side. It appears that every element of the set $A \cup B$ is accounted for once on the right-hand side of the equality. It is obvious that any other element (which does not belong to A or B) is not accounted for on the right-hand side. The equality is proved. □

Problem 2.20. *How many natural numbers are there in the first thousand, which are not divisible by* 3 *and by* 5?

Solution. This is a typical problem that requires the application of the inclusion-exclusion principle (see the previous exercise). Let N_3 be the set of those numbers (in the first thousand) which are divisible by 3, and N_5 be the set of those numbers which are divisible by 5. Then $N_3 \cup N_5$ is the set of numbers divisible by 3 or by 5, and $N_3 \cap N_5$ is the set of numbers divisible by 3 and by 5 at the same time. The wanted number is $1000 - |N_3 \cup N_5|$. Thus, our main task is to determine the number $|N_3 \cup N_5|$, which is straightforward with the help of the inclusion-exclusion principle, as

$$|N_3 \cup N_5| = |N_3| + |N_5| - |N_3 \cap N_5|.$$

The right-hand side of the above equality contains summands of the same type, as N_3, N_5 and $N_3 \cap N_5$ are the sets of numbers which are divisible by $3, 5$ and 15. We have:

$$\begin{aligned}&|N_3| = \left[\tfrac{1000}{3}\right] = 333, \quad |N_5| = \tfrac{1000}{5} = 200,\\ &|N_3 \cap N_5| = \left[\tfrac{1000}{15}\right] = 66,\end{aligned}$$

$$|N_3 \cup N_5| = 333 + 200 - 66 = 467.$$

Therefore, there are 533 numbers of the stated type.

Problem 2.21. *There are* 26 *students studying German or French.* 18 *of them study German and* 19 *study French. How many students study both languages?*

Answer. 11.

Problem 2.22. *How many natural numbers are less than 196 and mutually prime with this number? (Natural numbers m and n are called mutually prime if they do not have common divisors except* 1*).*

Answer. 84.

Problem 2.23. *Let p and q be different prime numbers, and k and s be natural numbers. How many natural numbers are there, which are less than $p^k \cdot q^s$ and mutually prime with this number?*

Answer. $p^k q^s \cdot \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$.

Problem 2.24. *Prove that for any finite sets A, B and C, the following equality holds (the inclusion-exclusion principle for three sets):*

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.$$

(Compare the above equality with the equality for two sets (see problem 2.19)).

Problem 2.25. *How many natural numbers less than or equal to 1000 are there, which are not divisible by $3, 5$ and 7 at the same time?*

Answer. 457.

Problem 2.26. *Let p, q and r be different prime natural numbers, and m, n, k be natural numbers. How many natural numbers are there, which are less than $p^m q^n r^k$ and mutually prime with this number?*

Answer. $p^m q^n r^k \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)$.

Problem 2.27. *Let E be a superset. Prove that:*

1. $\bar{\emptyset} = E$;
2. $\bar{E} = \emptyset$;
3. $\overline{A \cup B} = \bar{A} \cap \bar{B}$;
4. $\overline{A \cap B} = \bar{A} \cup \bar{B}$;
5. $\bar{\bar{A}} = A$.

Proof. 1. $\bar{\emptyset} = E \backslash \emptyset = E$.

2. $\bar{E} = E \backslash E = \emptyset$.

3. Let $x \in \overline{A \cup B}$; then $x \notin A \cup B$. Then the definition of union yields that $x \notin A$ and $x \notin B$. Thus, according to the definition of intersection, $x \in \bar{A} \cap \bar{B}$. We have proved that under the assumption that x is an element of the set $\overline{A \cup B}$, x should necessarily be an element of the set $\bar{A} \cap \bar{B}$. This fact evidences that any element of the set $\overline{A \cup B}$ is an element of $\bar{A} \cap \bar{B}$, hence $\overline{A \cup B} \subset \bar{A} \cap \bar{B}$.

   Now, it suffices to prove that $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$, to complete the proof of the equality of the sets $\overline{A \cup B}$ and $\bar{A} \cap \bar{B}$. Let $t \in \bar{A} \cap \bar{B}$. By the definition of intersection, we get $t \in \bar{A}$ and $t \in \bar{B}$, hence $t \notin A$ and $t \notin B$. The latter means that $t \notin A \cup B$, which results in the inclusion $t \in \overline{A \cup B}$. Therefore, $\bar{A} \cap \bar{B} \subset \overline{A \cup B}$.

□

Problem 2.28. *Let A, B and C be the subsets of a superset E.*

Prove the equalities:

1. $A \backslash B = A \cap \bar{B}$;

2. $\overline{A \backslash B} = \bar{A} \cup B$;

3. $A \backslash (B \cap C) = (A \cap \bar{B}) \cup (A \cap \bar{C})$;

4. $A \backslash (B \cup C) = A \cap \bar{B} \cap \bar{C}$;

5. $A \backslash E = \emptyset$.

Proof. 1. Let $x \in A \backslash B$. Then $x \in A$ and $x \notin B$. As $x \notin B$, we have that $x \in \bar{B}$ (by virtue of the definition of complement). It appears that $x \in A$ and $x \in \bar{B}$, hence $x \in A \cap \bar{B}$.

Now, assume that $x \in A \cap \bar{B}$. Then $x \in A$ and $x \in \bar{B}$, that is $x \in A$ and $x \notin B$, which results in $x \in A \backslash B$ (by the definition of difference).

Thus, we have proved that $A \backslash B \subset A \cap \bar{B}$ and $A \cap \bar{B} \subset A \backslash B$, hence $A \backslash B = A \cap \bar{B}$.

2. The best proof of this equality is based on three equalities, namely:

$A \backslash B = A \cap \bar{B}$ (the first equality of this exercise);

$\overline{A \cap B} = \bar{A} \cup \bar{B}$ (see problem 2.27);

$A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$ (see problem 2.10).

We have: $A \backslash (B \cap C) = A \cap \overline{(B \cap C)} = A \cap (\bar{B} \cup \bar{C}) = (A \cap \bar{B}) \cup (A \cap \bar{C})$.

□

Problem 2.29. *How many different correspondence can be established between sets A and B, where $|A| = 2$ and $|B| = 3$? How many ways are there to map the set A into the set B? How many ways are there to map the set B into the set A?*

Answer. 2^6; 9; 8.

Problem 2.30. *Let A and B be finite sets having 3 and 4 elements respectively.*

1. How many ways are there to map A into B?

2. How many injective mappings from A to B are there?

3. How many ways are there to map B into A?

4. How many different correspondences are there between the sets A and B?

5. How many ways are there to map B into B?

6. How many of these mappings are bijective?

Answer.

1. 4^3;

2. $4 \cdot 3 \cdot 2$;

3. 3^4;

4. 2^{12};

5. 4^4;

6. $4 \cdot 3 \cdot 2 \cdot 1$.

Problem 2.31. *How many bijections of the set* $\{1, 2, 3, 4, 5, 6, 7, 8\}$ *on itself are there, with the images of odd and even numbers being odd and even respectively?*

Answer. $(4!)^2$. $(4! = 1 \cdot 2 \cdot 3 \cdot 4)$.

Problem 2.32. *How many mappings of the set* $\{a, b, c, d, e, f\}$ *to itself are there, where the images of the elements a and f are the same?*

Answer. 6^5.

Problem 2.33. *How many mappings of the set* $\{1,2,3,4,5,6,7,8,9\}$ *to itself are there, with the images of even digits being odd digits?*

Answer. $5^4 \cdot 9^5$.

Problem 2.34. *How many bijections of the set* $\{1,2,3,4,5,6,7,8,9\}$ *to itself are there, with the images of even digits being odd digits?*

Answer. $(5 \cdot 4 \cdot 3 \cdot 2)^2$.

Problem 2.35. *How many injections of the set* $\{a, b, c, d\}$ *to the set* $\{1,2,3,4,5\}$ *are there, with the image of a being less than the image of b?*

Answer. $\frac{1}{2}(5 \cdot 4 \cdot 3 \cdot 2)$.

Problem 2.36. *How many mappings of the set* $\{a, b, c, d\}$ *to the set* $\{1,2,3,4,5\}$ *are there, with the image of a being less than or equal to the image of b?*

Answer. $5^3 \cdot 3$.

Problem 2.37. *Let* $N_k = \{1, 2, 3, ..., k\}$. *The Cartesian product* $N_k \times N_k$ *consists of all possible pairs of numbers* $(x; y)$, *both components of which belong to* N_k. *How many pairs* $(x; y)$ *are there in the set* $N_k \times N_k$, *which:*

1. $x > y$?

2. $x < y$?

3. $x \geq y$?

4. $x + y \leq k$?

5. $x + y \leq k + 1$?

6. $x + y > k + 1$?

Answer.

1. $\frac{k(k-1)}{2}$;

2. $\frac{k(k-1)}{2}$;

3. $\frac{k(k+1)}{2}$;

4. $\frac{k(k-1)}{2}$;

5. $\frac{k(k+1)}{2}$;

6. $\frac{k(k-1)}{2}$.

Hint. Compose a square table of elements of the set $N_k \times N_k$, in which an element (pair) $(i;\, j)$ stands in the intersection of the i-th row and the j-the column.

Problem 2.38. *A mapping $\varphi : N_k \times N_k \to N$ (N_k is defined in the previous exercise) is defined by the following expression: : $\varphi((i;\, j)) = i + j$, where $(i;\, j) \in N_k \times N_k$. Determine the image of the set $N_k \times N_k$ under this mapping. How many elements are there in this image?*

Answer. $2k - 1$.

Problem 2.39. *A mapping $\varphi : N_k \times N_k \to Z$, $\quad N_k = \{1, 2, 3, ..., k\}$, is defined by the rule $\varphi((i;\, j)) = |i - j|$. Determine the image of the set $N_k \times N_k$ under this mapping. How many elements are there in this image?*

Answer. k.

Problem 2.40. *A set A contains m elements, and a set B contains n.*

1. *How many different correspondences are there between the sets A and B?*
2. *How many ways are there to map A into B?*
3. *How many injective mappings from A to B are there?*

Answer.

1. 2^{mn};
2. n^m;
3. *there are none if $m > n$; $n(n-1)(n-2)\cdots(n-(m-1))$ if $m \le n$.*

Solution. 1. Any correspondence between A and B is defined by a certain subset of the Cartesian product $A \times B$. There is a bijection between the correspondences $A\varphi B$ and the subsets of $A \times B$. Therefore, there are the same amounts of both. Having determined the number of subsets of $A \times B$, we will know the number of correspondences between the elements of A and B. In order to define the amount of subsets of $A \times B$, we first need to find out how many elements are there in this set. By the rule of product, there are mn elements, hence there are 2^{mn} subsets.

2. In order to construct a mapping of A to B, we need to assign an image in the set B to each element from A. There are n options for any element from A (as there are n elements in B), and the choice made for anyone element does not affect the options for other elements. In other words, images of different elements of A can be combined arbitrarily. This is a typical situation in which we are able to apply the rule of product. For each of m elements

of A, there are n options for the choice of image, hence n^m different mappings can be constructed.

3. Arrange elements of A in some order: first, second, third, and so on up to the m-th element. To construct an injection $\varphi : A \to B$, we need to assign an image in the set B to each element from A, and different elements of A should have different images. The latter requirement means that no injection is possible if B has less than m elements.

Let $n \geq m$. Let us imitate the process of creation of the injection $\varphi : A \to B$, to find how many injections are possible. For the first element of A there are n options to choose its image in B (any element of B is suitable). Regardless of the element chosen to be the image of the first element, there are $n-1$ options for the image of the second element of A (it can be any element from B, except for the one chosen before). We proceed in a similar fashion. Finally, for the m -th element of the set A we will have $n-(m-1)$ options disregarding the choices made before. By the rule of product, we conclude that there are $n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(m-1))$ injections.

Problem 2.41. *In the previous exercise, we intentionally did not ask about the number of surjections from A to B. It appears that finding the amount of surjections is much more complicated than the amount of bijections or injections. As it was demonstrated above, it is enough to apply the rule of product to determine the amount of bijections or injections. The situation is different when we deal with surjections. The answer can not be given by any simple closed formula. Instead, we need to deduce a rather complex recurrence relation, as the amount of surjections depends on two natural numbers – the number of elements of A and B.*

So, let A and B be sets of m and n elements respectively: $|A| = m$, $|B| = n$. Under a surjective mapping $\varphi : A \to B$, every element of B has a non-empty preimage, hence in order for at least one surjection $\varphi : A \to B$ to exist, we need $|A| \geq |B|$. Assuming this requirement is fulfilled (the set A has no less elements than the set B), prove the following recursive formula for the amount of surjections:

$$d(m,n) = n\cdot(d(m-1,n)+d(m-1,n-1)).$$

Here, $d(s;t)$ denotes the amount of surjections of an s-element set to a t-element set. This notation is correct if $s<t$ as well, because we put $d(s,t)=0$ in this case

Solution. Choose an element a from the set A. We split all possible surjections $\varphi : A \to B$ into two groups. The first contains all those surjections, which do not have the one-element set $\{a\}$ among the preimages of the elements of B. The other surjections fall in the second group. They are those surjections, under which $\{a\}$ is a preimage of some element of B. Let us present this classification from another angle. If φ a surjection from A to B and $\varphi(a) = u$, $\varphi(c) = u$, then we will call the elements c and a related (with respect to the surjection φ). Thus, the elements c and a are related w.r.t. φ if both have the same image under φ. Now, we classify φ as a representative of the first group if the element a has at least one related element w.r.t. φ. If there are no elements relate to a, then the surjection φ belongs to the second group.

If we are able to express $d(s;t)$ through the amount of surjections in the first and the second group, then we will have the problem solved. First, we have to make one remark.

Under every surjection $\varphi : A \to B$, the elements of the set A are split into classes depending on their image in B. One class contains all related elements, that is those which have the same image. There are n classes (as the number of elements in B). Each class may contain one element or more elements that are related.

Let $\varphi : A \to B$ be a surjection from the first group. The element a is not the only element in its class under the above classification. Therefore, removing it, we get a surjection $\varphi_a : (A\backslash\{a\}) \to B$, under which all other elements have the same images as before. Consider an arbitrary bijection $\psi : (A\backslash\{a\}) \to B$. Adding the element a to any class of the partition of the set $A\backslash\{a\}$ under this surjection and assigning the corresponding image in B to it, we get the surjection from the set A to B. Obviously, this surjection belongs to the first group with respect to the element a. Thus, the given surjection $\psi : (A\backslash\{a\}) \to B$ can be transformed into a surjection $\varphi : A \to B$ in n different ways (by the number of classes in the partition of the set $A\backslash\{a\}$ under the surjection ψ). The conclusion is that the amount of surjections $\varphi : A \to B$ in the first group is n times greater than the amount of surjections of an $(m-1)$-element set to an n-element set. In other words, the first group contains $n \cdot d(m-1;n)$ surjections.

Let us count the amount of surjections $\varphi : A \to B$ in the second group. Recall that this group includes those surjections under which the element a is "isolated", that is it has no related elements and it defines a separate class in the partition of the set A w.r.t. the mapping φ. So, let $\varphi : A \to B$ be such surjection, and $\varphi(a) = b$. Removing the element a from the set A and the element b from the set B, we get a surjection $\varphi_b : (A\backslash\{a\}) \to (B\{b\})$. Conversely, if $\varphi_c : (A\backslash\{a\} \to B\backslash\{c\})$ is a surjection, then adding the element a to the first set and the element c to the second, and letting $\varphi(a) = c$, we get a surjection $\varphi : A \to B$, which belongs to the second group. As there are n different values for c (by the number of elements in B), the second groups contains n times more surjections $\varphi : A \to B$ that there are surjections of a $(m-1)$-element set to $(n-1)$-element set. Thus, the second group contains $n \cdot d(m-1;n-1)$ surjections. The recursive formula

$$d(m;n) = n \cdot (d(m-1;n) + d(m-1;n-1))$$

is proved.

Problem 2.42. *The recursive formula from the previous exercise is similar to the recursive formula defining Pascal's triangle. The numbers $d(m;n)$ depend on the values of m and n. Thus, they can be conveniently placed on a two-dimensional table 2.1. We construct the table, in which the values of m and n enumerate rows and columns respectively.*

Having filled in the first row and column, we are enabled to use the recursive formula. These can be filled in according to the following obvious equalities:

$d(m;1) = 1$ *and* $d(1;n) = 0$ *for* $n > 2$.

After this, the table can be filled in row by row with the help of the recursive formula. We suggest the reader extend the table up to the 8×8 *size.*

1. *The diagonal of the table stretching from the top left corner consists of numbers $d(m;n)$, $m = 1, 2, 3, \ldots$. Determine the direct formula for this sequence.*

2. *Take a closer look at the sequence of numbers in the second column of the table. Guess the law of this sequence. Find and prove the direct formula for this sequence*

Table 2.1. Values of $d(m; n)$.

$m\backslash n$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2	0	0	0	0
3	1	6	6	0	0	0
4	1	14	36	24	0	0
5	1	30	150	240	120	0
6	1	62	540	1560	1800	720

using on the recursive formula for $d(m; n)$. Then deduce it from scratch without using the recursive formula.

Answer.

1. $d(m; n) = m!$ *The number $d(m; n)$ defines the amount of bijections between two m-element sets.*

2. $2^n - 2$.

Solution.
The sequence of numbers t_m in the second column is defined by the initial condition $t_1 = 0$ and the recurrence relation

$$t_m = 2t_{m-1} + 2.$$

Indeed, this column contains the numbers $d(m; 2)$, $m = 1, 2, 3, \ldots$. Denoting $d(m; 2)$ by t_m for our convenience, we have:

$t_1 = d(1; 2) = 0;$
$t_m = d(m; 2) = 2(d(m-1;\ 2) + d(m-1;\ 1) = 2(d(m-1;\ 2) + 1) = 2t_{m-1} + 2.$

Observing six initial terms of the sequence
$0, 2, 6, 14, 30, 62,$
we see that each of them is less by 2 than the elements of the geometric progression
$2, 4, 8, 16, 32.64.$
We have strong basis for the following hypothesis:

$$t_m = 2^m - 2.$$

It remains to ensure that the recurrence relation holds for this formula. We have:

$$2t_{m-1} + 2 = 2 \cdot (2^{m-1} - 2) + 2 = 2^m - 2 = t_m.$$

Therefore, the direct formula $t_m = 2^m - 2$ really defines the sequence $t_m = d(m; 2)$.

There is also a combinatorial way to deduce the formula. Let a set B contains elements b_1 and b_2. A surjection $\varphi : A \to B$ is uniquely defined by the preimage of the element b_1, that is proper subset of the set A. There are $2^m - 2$ such subsets.

Problem 2.43. *A set A contains $2n$ elements, and a set B contains n elements. How many different surjections $\varphi : A \to B$ are there, under which any element of B has two preimages?*

Answer. $\frac{(2n)!}{2^n}$.

Hint. Let $B = \{b_1, b_2, ..., b_n\}$. We can imagine the process of creation of surjections $\varphi : A \to B$, which are the subject of the problem as follows. First, we choose the preimage of the element b_1 (C_{2n}^2 options); then we choose the preimage of b_2 (C_{2n-2}^2 options); then the same for b_3 (C_{2n-4}^2 options) and so on up to b_n (C_2^2 options). Applying the rule of product we get the answer.

Problem 2.44. *Find from scratch (without using previously deduced recursive formula) the amount of surjections $\varphi : A \to B$, if $|A| = m+1$, $|B| = m$.*

Answer. $C_m^2 \cdot m!$.

Problem 2.45. *Let $|A| = kn$, $|B| = n$. How many surjections $\varphi : A \to B$ are there, under which preimages of all elements of B consist of k elements?*

Answer. $\frac{(kn)!}{(k!)^n}$.

Problem 2.46. *Let $|A| = m$, $B = \{b_1, b_2, ..., b_n\}$, $s_1 + s_2 + ... + s_n = m$, where $s_1, s_2, ..., s_n$ are natural numbers. Find the amount of surjections $\varphi : A \to B$, under which $\left|\varphi^{-1}(b_1)\right| = s_1$, $\left|\varphi^{-1}(b_2)\right| = s_2$, $\ldots$, $\left|\varphi^{-1}(b_n)\right| = s_n$.*

Answer. $\frac{m!}{s_1!s_2!...s_n!}$.

Chapter 3

Basic Combinatorial Structures

1. Order. Permutations

Three objects, a, b and c can be placed in a straight line next to each other in six different ways. These are:

$$abc, \; acb, \; bac, \; bca, \; cab, \; cba.$$

How many ways are there to line up n objects?

Before answering this question, we provide a brief overview of appropriate terminology.

Let us have a finite set A containing n elements. We say that there is an ordered arrangement of A (or, equivalently, that A is arranged) if its elements are lined up one after another: first element, second, etc. In other words: to arrange an ordering in the set A means to enumerate its elements with numbers from 1 to n. Both definitions are equivalent. Indeed, lining up the elements of a set, we automatically establish their order: the initial element of the line becomes the first element, the next one is the second, and so on. For example, in the string bca the element b gets number 1, c is the second, and a is the third.

If there is an ordering (one of many possible) arranged on a set A, then this set is called an ordered set (concerning this exact ordering).

It is obvious that if A has more than one element, then there are multiple ways to arrange the order in it. What follows is one of the most important combinatorial questions:

How many ways are there to arrange a set of n elements?

A finite ordered set is also called a permutation (of corresponding elements). We emphasize that the word "permutation" is used here to define the object, which is a set of elements arranged in a certain way, and not the process of changing an ordered set. Using the introduced term, we can reformulate the above problem:

How many different permutations of n elements are there?

For example, there are two permutations of two elements a and b: ab and ba. There are six permutations of three elements, a, b and c: abc, acb, bac, bca, cab and cba.

Denote the wanted amount by P_n. This is a generally accepted notation: capital letter P with index n. Index denotes the number of elements (symbols, objects, etc.) participating in the creation of permutation. Thus, the symbol P_n is the number of different permutations, which can be made of n available elements. This is the combinatorial sense of the symbol

P_n. As we have seen above, this sense can be expressed in a slightly different form: P_n is the number of different ways to arrange a set of n elements in a straight line.

It remains to derive a formula for P_n. Let A be a set of n elements. We begin the process of arranging the order in it with the choice of the first element. Clearly, there are n ways to choose it (it can be any element of A). Once we have chosen it, the problem reduces to arranging of $n-1$ remaining elements. Using the introduced above notation, there are P_{n-1} ways to arrange $n-1$ elements. Therefore,

$$P_n = n \cdot P_{n-1}.$$

We have one-step recurrence relation for the numbers P_n. Accompanied by the initial condition $P_1 = 1$, it fully defines these numbers. It is also straightforward to derive the direct formula. Construct a descending chain of formulas, applying the recurrence relation to all smaller values of the index:

$$\begin{aligned}
&P_n = nP_{n-1},\\
&P_{n-1} = (n-1)P_{n-2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&P_3 = 3 \cdot P_2,\\
&P_2 = 2 \cdot P_1,\\
&P_1 = 1.
\end{aligned}$$

Multiplying these equalities term-wise, we derive

$$P_n = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

The product of all natural numbers from 1 to n is denoted by $n!$. Finally, we have:

$$P_n = n!$$

In particular,

$$P_2 = 2! = 1 \cdot 2 = 2,\ \ P_3 = 3! = 1 \cdot 2 \cdot 3 = 6,\ \ P_4 = 4! = 24.$$

The number P_n can be interpreted in another way. To create a permutation of given elements is to enumerate these elements with numbers from 1 to n. An enumeration is essentially the process of establishing a bijection between the set of n given elements and the set of natural numbers $\{1, 2, 3, \ldots, n\}$. Thus, the number P_n defines the amount of ways to establish a bijection between an n-element set A and the set $\{1, 2, 3, \ldots, n\}$. As we know, the amount of possible bijections does not depend on the nature of elements of the set, and it only depends on their amount. If we replace the set $\{1, 2, 3, \ldots, n\}$ with any other n-element set B, then we can state that P_n is the number of ways to establish a bijection between two n-element sets (A and B).

It is worth mentioning that in addition to the recurrence approach, the computational formula for the number of permutations P_n can be derived by application of the combinatorial rule of product. Really, assume there is an n-element set A. To enumerate it, we need to line up its elements. First, we choose the leftmost (first) element. There are n options to choose it. Regardless of the choice made for the first element, there are $n-1$ possibilities for the next one. There are $n-2$ options for the third element, and this amount of options

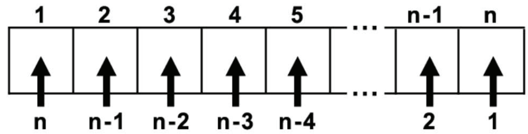

Figure 3.1. Permutations.

does not depend on the choice of two initial elements. We proceed similarly up to the last element. By this time there will be only one unused element, so there is no choice at the last step.

As we can see, the process of creation of permutation (or the process of ordering (arranging) of an n-element set) is completely compatible with the rule of product. To create a permutation means to fill in the string of n squares with the elements of the set A, and no element should appear in the squares twice. If we fill in the elements from the leftmost square to the rightmost then we have the following amounts of options to choose from on each step:

In addition, the number denoting the scope of possible choices in the k-th cell (where k is an arbitrary natural number from 1 to n) does not depend on the previous cells. This is the type of situation where the combinatorial rule of product can be applied. Therefore,

$$P_n = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Example 3.1. *There is a square table containing n^2 symbols (n symbols in each row and column). One needs to underline n symbols appearing once in each row and each column. How many ways to make this are there?*

We suggest the "chess-style" formulation of the above problem for those readers who prefer practical tasks:

How many ways are there to place n rooks of the same color on a chessboard, with no rooks attacking each other?

Rooks attack each other if they stand in the same row or column of squares (a horizontal or vertical string of squares) on a chessboard. Thus, the requirement is fulfilled if there is only one rook in each column and row.

As we usually do in such situations, imagine that we place n rooks on a chessboard one after another, ensuring that they do not attack each other. We begin with the first row of cells. It should necessarily contain exactly one rook. It can be placed in any of the n cells in the row. Let us have made our choice and have placed a rook in one of the cells of the first row. The second rook should be put in the second row because it should contain a rook as well. How many cells of the second row are there to choose from? There are $n-1$. The only forbidden cell is the one, which belongs to the column containing the first rook. Thus, we place the second rook in one of $n-1$ allowable cells of the second row of the chessboard. Now we have two rooks having two columns in their "firing range", so there are $n-2$ permitted cells in the third row for the third rook. Note that the amount of options does not depend on the actual placing of the first two rooks in the first two rows. We are in control of the situation now and we can forecast the further development correctly and

confidently up to the n-th rook. Placing rooks one after another, we "disable" one column on every step, thus reducing the freedom of choice of cells on the next row for the next rook by 1. For the penultimate, $(n-1)$-th rook there are two permitted cells in the $(n-1)$-th row, and for the last one, there is the only possible place in the last row. By virtue of the combinatorial rule of product, we conclude that there are $n!$ ways to place rooks.

As we can see, the answer coincides with the number of attainable permutations of n elements. This means that the problem of rooks' placement is the "twin" of the problem about permutations. There should be a "natural" bijection between the placement of rooks on a $n \times n$ chessboard and the permutations of n symbols.

Let us enumerate the rows and columns of the chessboard with numbers $1, 2, 3, ..., n$. Then the cells (fields, squares) of the chessboard get "double numbers" as their names, that is their names are coming from the set $N_n \times N_n$, where $N_n = \{1, 2, 3, ..., n\}$.

Assume that n rooks are placed on the chessboard, with exactly one rook standing in each row and column. Their places (cells) can now be addressed by their arithmetical names – the pairs of coordinates. If we name the rows in the usual manner (first, second, third, etc.), then the information about the positioning of the rooks is given by the following chain of pairs of numbers:

$$(1; i_1),\ (2; i_2),\ (3; i_3),\ ...,\ (n-1; i_{n-1}),\ (n; i_n). \tag{3.1}$$

The second components of pairs form some permutation of numbers $1, 2, 3, ..., n$ (as one rook stands in each column). Conversely, every permutation $(i_1, i_2, i_3, ..., i_n)$ of numbers $1, 2, ..., n$ have a certain placing of rooks on the chessboard corresponding to it. The chain of pairs (3.1) defines this placing. Hence, there is indeed a bijection between the permutations of n numbers and possible placings of rooks on the chessboard.

Example 3.2. *How many permutations of n symbols begin with given k symbols standing in predefined order?*

Let us illustrate the problem. Put $n = 5$. Let the problem be about those permutations of symbols (letters) a, b, c, d, e, which begin with the letters b and c, standing in this exact order. Here is the complete list of such permutations

$$\begin{array}{ll} bcade & bcdea \\ bcaed & bcead \\ bcdae & bceda \end{array}$$

It is clear why there are six of them. This is the amount of permutations that can be created with symbols a, d, e. The initial symbols b and c do not affect the amount of permutations, as they appear to be a fixed "add-on" to the variables in the last three positions.

The situation is absolutely similar in the general case. As the initial k symbols do not change from permutation to permutation, they do not affect the amount of those. Permutations differ only in their $n-k$ symbols. Therefore, there are $(n-k)!$ permutations.

2. Tuples

Let A be a set consisting of any elements (objects, symbols, etc). A tuple of elements of A of length k is a sequence (chain, ordered list) of k elements of A. A tuple of length k is also

called a k-tuple. According to the definition, two k-tuples can differ from each other by the elements forming them or by their order.

In order to illustrate the above definition we provide the full list of 2-tuples and 3-tuples, created of the elements of the set

$$\{a,b,c,d\}.$$

The 2-tuples are:

$$ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc.$$

The list of 3-tuples is the following:

$$abc, acb, bac, bca, cab, cba, abd, adb, bad, dba, dab, dba,$$

$$acd, adc, cad, cda, dab, dba, bcd, bdc, cbd, cdb, dbc, dcb.$$

The tuples of length 4, are essentially the permutations of the elements of the given set. There are $4!$ of them, so the answer is 24.

The length of a tuple can not exceed the number of elements of the set. Therefore, there are no tuples of length 5 or more consisting of the elements of the set $\{a,b,c,d\}$.

Let a set A contain n elements. How many k-tuples of elements of A can be created? This is a fundamental combinatorial question in the context of tuples. Although fundamentality does not necessarily imply complexity. There are multiple ways to find the answer. We outline two of them.

Approach I. First, we draw a template for a k-tuple, which is a string split into k squares: first, second, third and so on up to k-th. Filling the elements of A into squares, with different elements being placed in different squares, we get a k-tuple. To find out how many ways are there to construct a tuple, imagine the process of filling in squares one by one with elements of A. There are n candidates for the first square (any element of A), $(n-1)$ for the second (any element of A, except for the one selected for the first square) and so on. The amount of possible options decreases by one with each next square disregarding the actual choices made for the previous squares. There will be $n-(k-1)$ options for the last square. All necessary attributes for the application of the rule of product are available. Hence,

$$n(n-1)(n-2)...(n-(k-1))$$

tuples of length k can be created with n elements. Obviously, this result takes place for natural values of k from the interval $[1,n]$. If $k > n$, then no k-tuples exist.

Introduce the following notation: the number of k-tuples, which can be created with n elements, is denoted by the symbol A_n^k. Thus, we have derived a direct computational formula for this amount:

$$A_n^k = n(n-1)(n-2)...(n-(k-1)).$$

For example,

$$A_4^2 = 4\cdot 3 = 12,\ A_4^3 = 4\cdot 3\cdot 2 = 24,\ A_7^3 = 7\cdot 6\cdot 5 = 210,\ A_{10}^4 = 10\cdot 9\cdot 8\cdot 7 = 5040.$$

Approach II. Assume that different permutations of the elements of A are created. Remove the last $n-k$ elements of every permutation, preserving k initial elements. We get

k-tuples created of the elements of the set A. Each of them repeats $(n-k)!$ times, which is the result derived in the second example of the previous section. This is the amount of permutations beginning with a given k-tuple. We conclude that there are $\frac{P_n}{(n-k)!}$ k-tuples, which is in concordance with the result derived with the first approach. Hence,

$$A_n^k = \frac{P_n}{(n-k)!} = \frac{n!}{(n-k)!}.$$

Example 3.3. *How many five-digit numbers are there, with all their digits being different and not equal to zero?*

This number can be interpreted as a 5-tuple, constructed with digits $1,2,3,4,5,6,7,8,9$. Hence, the answer is

$$A_9^5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120.$$

Example 3.4. *How many five-digit numbers are there, with all their digits being different?*

Here we deal with those 5-tuples composed of ten digits that do not begin with zero (a five-digit number can not begin with zero). There are A_{10}^5 5-tuples and A_9^4 of them begin with zero. Therefore, there are

$$A_{10}^5 - A_9^4 = 27216$$

numbers of interest. The amount of wanted numbers can also be calculated by direct application of the rule of product. The first digit can be any digit except for zero (9 options), the second can be any (including zero), except for the one chosen for the first position (9 options), the third can be any digit except for the two initial (8 options), the fourth can be any except for the three initial (7 options), and finally, the fifth can be any except for the four initial (6 options). According to the rule of product, there are $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ wanted numbers.

3. Subsets

Let a set A contains n elements. Let us fix a number k and ask ourselves: how many k-element subsets of the set A are there? In this section, we find the answer to this question.

To put it another way, we want to find the number of ways to choose k elements out of n available.

The sought number is denoted by the symbol C_n^k. This notation is widely used, though not the only available. Another usual notation is $\binom{k}{n}$.

If a set A contains n elements, then its subsets can be divided into $n+1$ classes according to their cardinality (the number of elements they contain): an empty set, one-element subsets, two-element subsets, three-element subsets, and so on up to the n-element subset. The latter coincides with the whole set A. For instance, we provide the classified list of subsets of the set $\{a,b,c,d,e\}$:

One empty set $\emptyset$.

Five one-element subsets:

$$\{a\},\{b\},\{c,\}\{d\},\{e\}.$$

Ten two-element subsets:

$$\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\},\{b,e\},\{c,d\},\{c,e\},\{d,e\}.$$

Ten three-element subsets:

$$\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,d\},\{a,c,e\},$$

$$\{a,d,e\},\{b,c,d\},\{b,c,e\},\{b,d,e\},\{c,d,e\}.$$

Five four-element subsets:

$$\{a,b,c,d\},\{a,b,c,e\},\{a,b,d,e\},\{a,c,d,e\},\{b,c,d,e\}.$$

Finally, there is one five-element subset:

$$\{a,b,c,d,e\}.$$

We need to learn how to calculate the amount of subsets in each class, that is to determine the value of the symbol C_n^k for any n and k ($0 \leq k \leq n$).

The numbers C_n^k play an important role in combinatorics. Therefore, we derive the direct formula for it in three different ways.

First Approach. The computational formula for C_n^k can be derived based on the computational formula for A_n^k, deduced in the previous section. Tuples of length k, consisting of the elements of A are nothing else than ordered k-element subsets of this set. The number A_n^k denotes the amount of such subsets. Let us split all ordered k-subsets of the set A (which are k-tuples) into groups according to their cardinality. Every group will contain all those tuples which consist of the same elements that are ordered differently. Now, we have to answer two questions.

The first question is: how many tuples are there in each group?

The second question is: how many such groups are there?

We are prepared to answer both questions. Tuples in each group differ by the order of their elements only. The elements composing the tuples of the group are the same. Therefore, all of them are the permutations of the same k elements. The group contains all such permutations, as according to the terms it includes all tuples of given k elements. Thus, the group contains $k!$ tuples. This conclusion is correct for any group because the above considerations apply to the arbitrary group.

Thus, any group consists of $k!$ tuples. And how many groups are there? Every group corresponds to one of k-subsets of the set A. Therefore, the number of groups is the same as the number of k-element subsets of the set A, which is C_n^k (according to the introduced notation). This means that the number $k!{\cdot}C_n^k$ denotes the amount of all k-tuples that can be created out of the elements of the set A. On the other hand, this amount is expressed by the number A_n^k, hence

$$k! \cdot C_n^k = A_n^k.$$

As a result, we get the computational formula for C_n^k:

$$C_n^k = \frac{1}{k!}A_n^k = \frac{n(n-1)(n-2)\ldots(n-(k-1))}{k!}.$$

For example,

$$C_5^2 = \frac{5 \cdot 4}{2!} = 10, \quad C_7^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

Note, that for $k = n$ the formula for C_n^k yields 1, as required. In addition, we require that the formula provides the correct result for $k = 0$. In order for this to be true, the equality $0! = 1$ is assumed to hold, and multiplying the numerator and the denominator by $(n-k)!$ of the above formula, we express it as follows:

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Second Approach. Let A be a set of n elements. We want to derive the formula answering the question: how many k-element subsets of the set A are there? We denote the number of interest by the symbol C_n^k. Thus, we are talking about the computational formula for this number. First, we will try to find the connection between the numbers C_n^k and C_{n-1}^{k-1} (assuming $k > 0$ and $k \leq n$). To this end, we consider the pairs $(b; B)$, where B is a k-element subset of the set A, and b is its element. How many such pairs are there? We provide the answer to this question using the symbols C_n^k and C_{n-1}^{k-1} as if they are known numbers. Moreover, we will answer this question in two ways (providing two different formulas).

A pair $(b; B)$ can be created as follows. First, we choose the subset B (there are C_n^k ways to make it). Then we choose some element b of B (k ways). By virtue of the combinatorial rule of product, we get that there are $C_n^k \cdot k$ pairs $(b; B)$. However, a pair $(b; B)$ can be constructed in a different way. First, we choose some element b from the set A (n options), and then choose a $(k-1)$-element subset D of $A \setminus \{b\}$ (C_{n-1}^{k-1} options). Adding the element b to this subset we get a k-element subset B. It appears that there are $n \cdot C_{n-1}^{k-1}$ pairs $(b; B)$ this time. As the number of pairs is correct in both cases, the results should coincide. Hence the equality

$$C_n^k \cdot k = n \cdot C_{n-1}^{k-1}$$

holds, and

$$C_n^k = \frac{n}{k} C_{n-1}^{k-1}.$$

We derived the recurrence relation for C_n^k. What differs it from recurrence relations, which we have encountered before, is that the "descent" occurs by two indices: n and k. This difference is insignificant. As before, we can construct a descending chain of equalities that ends when the upper index becomes 0. The equality $C_m^0 = 1$, which is correct for any integer non-negative m, plays the role of the initial condition. Thus we get:

$$C_n^k = \frac{n}{k} C_{n-1}^{k-1},$$

$$C_{n-1}^{k-1} = \tfrac{n-1}{k-1} C_{n-2}^{k-2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$C_{n-(k-2)}^{2} = \tfrac{n-(k-2)}{2} C_{n-(k-1)}^{1},$$

$$C_{n-(k-1)}^{1} = \tfrac{n-(k-1)}{1} C_{n-k}^{0},$$

$$C_{n-k}^{0} = 1.$$

Multiplying these equalities term-wise, we get the direct formula for C_n^k:

$$C_n^k = \frac{n(n-1)(n-2)\ldots(n-(k-1))}{k!}. \tag{3.2}$$

Third Approach. Let us split all permutations of an n-element set A into classes (groups) according to the k-element sets formed by their k initial terms. Every class contains all permutations, k initial elements of which form the same k-element sets. For instance, consider the 7-element set $\{a, b, c, d, e, f, g\}$ and $k = 3$. The permutations

$$adfcbge,\ afdcgeb,\ afdgebc,\ dafbec,\ dafbgce,\ fdaecbg,\cdots$$

belong to the same class, because three initial elements of them form the same set $\{a, d, f\}$. Turning back to the general case, we ask ourselves: how many permutations are there in one class? It is not hard to answer this question. Consider an arbitrary permutation. What changes can be made to it, so that it remains in its class? Obviously, we can freely change the positions of its k initial elements, and change its $n-k$ last elements. Overall, there are $k!(n-k)!$ different permutations. Any other change of order of its elements results in permutation being moved to another class.

It appears that each class contains $k!(n-k)!$ permutations. We conclude that there are $\frac{n!}{k!(n-k)!}$ classes. But there is a bijection between these classes and k-element subsets of the n-element set A (according to how the classes have been constructed). Therefore, there are as many subsets, as there are classes, that is

$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{3.3}$$

4. Numbers C_n^k: Combinatorial and Computational Aspects

As it has been discussed above, C_n^k is the number denoting the amount of different k-element subsets that can be constructed with the elements of an n-element set.

The combinatorial sense of the numbers C_n^k can be expressed in a slightly different way. The number C_n^k denotes the amount of ways to choose k objects out of n available. Although in the first case we talk about the list of certain objects (k-element subsets), and in the second case there is a single action of choice, it is obvious that definitions are equivalent.

The numbers C_n^k are very popular in combinatorics. There is an excessive amount of combinatorial problems (including theoretically important) the solutions of which are given by the numbers C_n^k. In certain sense, these numbers are combinatorial constants, manipulating with which one often (though, obviously not always) can express the sought amount.

In formula (3.3), if we put $m = n-k$ for C_n^k, then it gets the symmetrical form with respect to k and m:

$$C_n^k = \frac{n!}{k!m!} = C_n^m.$$

This form is convenient for theoretical investigation of C_n^k. It shows how we can act when we have to compute C_n^k. As $C_n^k = C_n^m$ (where $k+m=n$), direct formula (3.2) can be applied

to the smallest of the numbers k or m. For example, if we are about to compute C_{10}^{3}, then we directly apply formula (3.2):

$$C_{10}^{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120.$$

On the other hand, if we need to find C_{11}^{8}, then we first use the equality $C_{11}^{8} = C_{11}^{3}$ and then formula (3.2):

$$C_{11}^{8} = C_{11}^{3} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165.$$

Problems

Problem 3.1. *How many six-digit numbers consist of six different digits, among which there is no zero?* *Answer.* A_9^6.

Problem 3.2. *How many four-digit numbers have their digits in decreasing order? (such as* 8420, 7651, 4210, 9543 *etc.).*

Answer. C_{10}^{4}.

Problem 3.3. *How many four-digit numbers have their digits in increasing order? (such as* 1289, 2458, 3789, 1468 *etc.).*

Answer. C_9^4.

Problem 3.4. *How many five-digit numbers have all their digits odd and different?*

Answer. P_5.

Problem 3.5. *How many five-digit numbers have all their digits even and different?*

Answer. 96.

Problem 3.6. *Below, we use the word "chessboard" to denote a square board split into smaller squares (fields, cells of a chessboard).*

1. *How many ways are there to place two black rooks and three black pawns on a* 4×4 *chessboard?*
2. *How many ways are there to place eight black and eight white pawns on a* 4×4 *chessboard?*
3. *How many ways are there to place two rooks of different colors and three black pawns on a* 4×4 *chessboard?*
4. *How many ways are there to place two white and two black rooks on a* 4×4 *chessboard so that they do not attack each other? (that is, exactly one rook should stand in each row and column of cells)?*

Answer.

1. $C_{16}^{2} \cdot C_{14}^{3}$;
2. C_{16}^{8};

3. $C_{16}^1 \cdot C_{15}^1 \cdot C_{14}^3$;

4. $4! \cdot C_4^2$.

Problem 3.7. *There are n lines on the plane, any two of which intersect and any three do not have common point.*

1. *How many points of intersection are there?*

2. *How many triangles are there, with all their sides lying on the lines?*

3. *How many triangles have their vertices in the points of intersection?*

Answer.

1. C_n^2;

2. C_n^3;

3. $C_{C_n^2}^3 - n \cdot C_{n-1}^3$ $(n > 3)$.

Problem 3.8. *Sets A and B contain n elements each. How many ways are there to establish a bijection between A and B?*

Answer. $n!$.

Problem 3.9. *A set A contains n elements. c is one of its elements.*

1. *How many k-element subsets of the set A include the element c?*

2. *How many k-element subsets of the set A do not include the element c?*

3. *How many subsets of the set A do not include the element c?*

4. *How many subsets of the set A include the element c?*

Answer.

1. C_{n-1}^{k-1};

2. C_{n-1}^k;

3. 2^{n-1};

4. 2^{n-1}.

Problem 3.10. *How many permutations of the numbers* $1, 2, 3, 4, \ldots, n-1$, n *are there, with the numbers 1 and 2 standing next to each other?*

Answer. $2 \cdot (n-1)!$.

Problem 3.11. *How many permutations of* $2n$ *initial natural numbers are there, where:*

1. *Odd and even numbers alternate?*

2. *Odd numbers occupy n initial positions?*

3. *The first number is odd and the last is even?*

4. *The first and the last numbers are even?*

5. *The sum of the first and the last numbers makes an even number?*

6. *Odd numbers are in ascending order, as well as even numbers?*

Answer.

1. $2 \cdot (n!)^2$;

2. $(n!)^2$;

3. $n^2 \cdot (2n-2)!$;

4. $n(n-1) \cdot (2n-2)!$;

5. $2n(n-1) \cdot (2n-2)!$;

6. C_{2n}^n. *Hint.* 6) *In order to create such permutation, it is enough to list n places, in which even numbers stand.*

Problem 3.12. *How many permutations of the set of numbers* $1, 2, 3, \ldots, n-1, n$ *have numbers 1 and 2 separated from each other by k other numbers* $(0 \leq k \leq n-2)$*?*
Answer. $2 \cdot (n-k-1) \cdot (n-2)!$.

Problem 3.13. *How many permutations of the set of numbers* $1, 2, 3, \ldots, n-1, n$ *begin with number 1 and end with number n?*
Answer. $(n^2-3n+3) \cdot (n-2)!$.

Problem 3.14. *How many permutations of the set of numbers* $1, 2, 3, \ldots, n-1, n$ *have number 1 in one of the initial four positions and number 2 in fifth position or further?*
Answer. $4 \cdot (n-4) \cdot (n-2)!$.

Problem 3.15. *There are n points on a circle, which we further call base points.*

1. *How many chords are bounded with the base points?*

2. *How many triangles have their vertices in the base points?*

3. *How many quadrilaterals have their vertices in the base points?*

Answer.

1. C_n^2;

2. C_n^3;

3. C_n^4.

Problem 3.16. *There are n points on one side of a triangle, m points on another, and k points on the third side. None of the points coincide with the vertices of the triangle.*

1. *How many triangles have all their vertices in some of these points?*

2. *How many quadrilaterals have all their vertices in some of these points?*

Answer.

1. $C^3_{n+m+k} - C^3_n - C^3_m - C^3_k$;
2. $C^4_{n+m+k} - C^4_n - C^4_m - C^4_k - (m+k)C^3_n - (k+n)C^3_m - (n+m)C^3_k$.

Problem 3.17. *Three parallel lines l_1, l_2 and l_3 do not belong to the same plane. There are n points on l_1, m on l_2 and k on l_3.*

1. *How many triangles have all their vertices in some of these points?*

2. *How many tetrahedra have all their vertices in some of these points?*

Answer.

1. $C^3_{n+m+k} - C^3_n - C^3_m - C^3_k$;
2. $mkC^2_n + knC^2_m + nmC^2_k$.

Problem 3.18.

There are n points on a circle. How many convex polygons (triangles, quadrilaterals, pentagons and all other) have all their vertices in some of these points?

Answer. $2^n - C^2_n - n - 1$.

Problem 3.19. *How many different triangles with sides of integer lengths from the interval $(n, 2n]$ (n is natural) are there? How many of them are isosceles but not equilateral? How many of these triangles are equilateral? How many triangles are scalene?*

Answer. There are C^3_n scalene triangles; $n(n-1)$ isosceles but not equilateral; n equilateral.

Hint. Every number from the interval $(n, 2n]$ is less than the sum of any two other numbers from this interval. Therefore, any triplet $a, b, c \in (n, 2n]$ defines a triangle with integer-valued sides.

Problem 3.20. *Let A be the set of triangles with sides of integer lengths from the interval $(p, 2p]$ (p is natural), and B be the set of scalene triangles with sides of integer lengths from the interval $(p, 2p+2]$. Establish a bijection between the sets A and B.*

Hint. Match the triangle from the set A with the sides a, b, c $(a \le b \le c)$ with the triangle from the set B with the sides $a,\ b+1,\ c+2$.

Problem 3.21. *There is a convex n-gon (a polygon with n sides) such that any three of its diagonals do not intersect in one point inside it.*

1. *How many points of intersection of its diagonals are there inside the n-gon?*

2. *How many triangles are formed with diagonals of the n-gon?*

3. *How many parts do the diagonals split the n-gon into?*

Answer.

1. C_n^4;

2. $\frac{1}{6}n(n-4)(n-5)$ $(n \geq 5)$;

3. $1+C_n^4+\frac{n(n-3)}{2}$.

Sketch of Solution.

1. Let M be the point of intersection of the diagonals PQ and RS. The points P, R, Q and S are the vertices of n-gon, which at the same time are the vertices of a convex quadrilateral. The sides of this quadrilateral are the sides or diagonals of the given n-gon. In the latter case, they do not intersect with each other and with the diagonals PQ and RS inside the n-gon. This implies that the point M can be naturally matched with four vertices P, R, Q, S of the given n-gon, and the correspondence between them is bijective. Thus every point of intersection of the diagonals corresponds to a certain quad of vertices, and any quad of vertices of the n-gon corresponds to a certain point of intersection.

2. First, we can count the triangles, the sides of which are diagonals or sides of the n-gon. It is straightforward to count them because they are in bijective correspondence with the triplets of vertices of the n-gon. Then, we have to subtract from the obtained number the amount of the triangles, some of the sides of which are the sides of the n-gon. There are two types of such triangles: those triangles that have two common sides with the n-gon, and those that share one side with it. There are n triangles of the first type, and $n(n-4)$ of the second.

3. Imagine that some (but not all) of the diagonals are drawn and they split the triangle into some (s) parts. We draw the new diagonal and attempt to find out how the number s changes and what affects the extent of its increase. Suppose that the new diagonal intersects the available diagonals in k points. These points split the new diagonal into $k+1$ intervals and every interval splits the previously solid part of the n-gon into two parts. Hence the number s increases by $k+1$ when the new diagonal is drawn. It appears that the number s increases by k (the amount of new points of intersection) plus 1 (the amount of new diagonals) upon the drawing of a new diagonal. This information is sufficient to solve the problem. It remains to take into account that before all diagonals the polygon was a solid part of the plane Imagine the drawing of diagonals one by one. After the last diagonal is drawn the initial amount of parts (1) increases by the sum of two numbers: the amount of diagonals and the amount of points of their intersection. The conclusion is that the diagonals split the n-gon into the following number of parts:

1+ (the number of diagonals)+(the number of points of intersection of the diagonals inside the n-gon).

The amount of diagonals may be counted as follows: C_n^2 is the amount of intervals connecting the vertices of the n-gon; n is the number of its sides; other intervals are its diagonals.

Problem 3.22. *How many ways are there to seat n people around a circular table where two seatings are considered the same when everyone has the same two neighbors with regard to whether they are right or left neighbors?*

Answer. $(n-1)!$.

Hint. Seating a person K in some place at the table, we break the closed chain. To seat other people means to create some permutation.

Problem 3.23. *16 cadets are about to line up:*

1. *in a single-file line;*
2. *in rows of two;*
3. *in rows of four. How many ways are there for them to make it?*

Answer.

1. *16!;*
2. *16!;*
3. *16!.*

Problem 3.24. *There are 16 students taking part in the prom night: 8 boys and 8 girls. The waltz is about to start. How many ways are there for the students to make pairs?*

Answer. *8!*

Problem 3.25. *There are 16 students in a class and 20 available seats. How many ways can the students be seated?*

Answer. A_{20}^{16}.

Problem 3.26. *There are 16 students in a class: 8 boys and 8 girls. They decided that there should be one girl and one boy seating at each of eight two-man desks. How many ways are there to seat the students?*

Answer. $2^8 \cdot (8!)^2$.

Problem 3.27. *A set A contains n elements, a and b are two of them. How many k-element subsets of the set A:*

1. *do not include neither a nor b?*
2. *include at least one of these elements?*
3. *include exactly one of these elements?*

Answer.

1. C_{n-2}^k;
2. $C_n^k - C_{n-2}^k$;
3. $2C_{n-2}^{k-1}$.

Problem 3.28. *Let A be an n-elements set. We call a chain of its subsets*

$$A_1, A_2, A_3, \ldots, A_n$$

an ascending dense chain if:

1. $|A_i| = i \ (i = 1 2, ..., n)$;

2. $A_i \subset A_{i+1} \ (i = 1, 2, ..., n-1)$;

3. $A_n = A$.

How many dense ascending chains does the set A have?

Answer. $n!$.

Hint. Establish a bijection between all permutations of the set A and all dense ascending chains of its subsets.

Problem 3.29. *Nine tourists decided to split into three groups of three people each. The first group will be responsible for getting firewood, the second will go to the village to buy milk and the third will prepare beds and dinner. How many ways are there for them to realize their plan?*

Answer. $C_9^3 \cdot C_6^3$.

Problem 3.30. *Nine tourists decided to split in triplets to sleep in three identical tents. How many ways can they form triplets?*

Answer. $\frac{1}{3!} C_9^3 \cdot C_6^3$.

Problem 3.31. *A subset B of a set A is called an invariant set with respect to a mapping* $\varphi : A \to A$ *if* $\varphi(B) \subset B$.

Let $|A| = n, \ B \subset A$ *and* $|B| = m$.

1. *How many mappings* $\varphi : A \to A$ *are there?*

2. *How many mappings* $\varphi : A \to A$ *exist, with respect to which the subset B is invariant?*

3. *How many mappings* $\varphi : A \to A$ *exist, with respect to which the subsets B and* $A \backslash B$ *are invariant?*

4. *Let*

$$A_1 \subset A_2 \subset A_3 \subset ... \subset A_n = A$$

be an ascending dense chain of subsets of the set A (see Exercise 28). How many mappings $\varphi : A \to A$ *are there, with respect to which all subsets of this chain are invariant?*

Answer.

1. n^n;

2. $m^m \cdot n^{n-m}$;

3. $m^m \cdot (n-m)^{n-m}$;

4. $n!$.

Problem 3.32. *1. Some of the natural solutions to the equation*

$$x+y+z=10$$

are: $(1;1;8),(1;8;1),(4;3;3),(5;2;3)$. *How many natural solutions to this equation are there?*

2. How many natural solutions to the equation

$$x+y+z=n$$

are there (n is a given natural number)?

3. How many natural solutions to the equation

$$x_1+x_2+x_3+...+x_k=n$$

are there, where $x_1, x_2, ..., x_k$ are unknown (n is a given natural number)?

Answer.

1. C_9^2;
2. C_{n-1}^2;
3. C_{n-1}^{k-1}.

Solution. 3 Clearly, there are no natural solutions if $n<k$. Assume $n \geq k$. Express the number n as the sum of ones:

$$1+1+1+1+...+1=n. \tag{3.4}$$

There are $n-1$ "+" signs in the left-hand side. Choosing $k-1$ "+" signs and "reducing" k sums of ones which are separated by the chosen signs, we get the expression of n as the sum of k natural number that forms the solution to the given equation. It is obvious that any solution can be obtained in this way. Therefore, there are C_{n-1}^{k-1} solutions to the equation, as this is the number of ways to choose $k-1$ of the available $n-1$ "+" signs in equality (3.4).

Problem 3.33. *How many integer non-negative solutions are there to the equation*

$$x_1+x_2+x_3+...+x_{k-1}+x_k=n$$

(the variables can get zero values as well as the natural ones)?

Answer. C_{n+k-1}^{k-1}.

Solution. Along with the equation

$$x_1+x_2+x_3+...+x_{k-1}+x_k=n, \tag{3.5}$$

we consider the equation

$$x_1+x_2+x_3+...+x_{k-1}+x_k=n+k. \tag{3.6}$$

Let us ensure that equation (3.5) has the same amount of integer non-negative solutions as the amount of natural solutions to equation (3.6). To this end, establish a bijection between both sets of solutions. Let

$$(a_1,\ a_2,\ a_3,\ \ldots,\ a_{k-1},\ a_k)$$

be an integer non-negative solution to equation (3.5). Then

$$(a_1+1,\ a_2+1,\ a_3+1,\ \ldots,\ a_{k-1}+1,\ a_k+1)$$

is a natural solution to equation (3.6) because:

a) all numbers a_i+1 are greater than or equal to 1 (as the numbers a_i are greater than or equal to zero);

b) their sum equals to $n+k$ (because the sum of $a_1,\ a_2,\ \ldots,\ a_k$ equals to n).

Conversely, if

$$(b_1,\ b_2,\ b_3,\ \ldots,\ b_{k-1},\ b_k)$$

is a natural solution to equation (3.6), then

$$(b_1-1,\ b_2-1,\ b_3-1,\ \ldots,\ b_{k-1}-1,\ b_k-1)$$

is an integer non-negative solution to equation (3.5). This results from:

a) the numbers $b_1-1,\ b_2-1,\ b_3-1,\ \ldots,\ b_k-1$ are greater than or equal to zero;

b) their sum is equal to n.

The above findings mean that there is indeed a bijection between integer non-negative solutions to the equation (3.5) and natural solutions to (3.6). In addition, this bijection is expressed by a simple rule:

If we add 1 to all components of all integer non-negative solutions to equation (3.5), we get all natural solutions to (3.6); if we subtract 1 from all components of natural solutions to equation (3.5), we get all integer non-negative solutions to (3.6). Therefore, there is the same amount of solutions to both equations, which equals to

$$C_{n+k-1}^{k-1},$$

as this is the amount of natural solutions to equation (3.6) according to the previous problem.

Problem 3.34. *How many integer non-negative solutions to the equation*

$$x_1+x_2+x_3+\ldots+x_{k-1}+x_k=n$$

have non-zero first component (the value of x_1 is non-zero)?

Answer. C_{n+k-2}^{k-1}.

Solution. Let A be the set of wanted solutions to the equation

$$x_1+x_2+x_3+\ldots+x_{k-1}+x_k=n,$$

and B be the set of all integer non-negative solutions to the equation

$$x_1+x_2+x_3+\ldots+x_{k-1}+x_k=n-1.$$

A bijection can be established between A and B: subtracting 1 from the first components of the solutions from A, we get the solutions from B; conversely, by increasing the first components of the solutions B by 1, we get the solutions from A. This yields that the sets A and B contain the same amount of solution. According to the previous exercise, the set B contains C_{n+k-2}^{k-1} solutions. Hence, there is the same amount of solutions in A.

Problem 3.35. *How many natural solutions are there to the inequality*

$$x_1+x_2+x_3+\ldots+x_{k-1}+x_k<n, \tag{3.7}$$

where x_1, x_2, x_3, ..., x_k are unknown?

Answer. C_{n-1}^{k}.

Solution. First, note that the above inequality does not always have solutions. We are dealing with the integer positive values of x_i. The smallest of them is 1. Even if all x_i have the smallest value, the inequality does not hold when n is less than $k+1$. The same is more than true for other natural values of unknowns. Hence the condition for the existence of solutions: $n \geq k+1$.

Assume this condition is satisfied and count the number of solutions to inequality (3.7).

First Approach. The first idea is as follows. If $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k)$ is a solution to inequality (3.7), then

$$\gamma_1+\gamma_2+\gamma_3+\ldots+\gamma_k=s<n,$$

and $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k)$ is a natural solution to the equation

$$x_1+x_2+x_3+\ldots+x_k=s. \tag{3.8}$$

Conversely, if $s<n$ and $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k)$ is a solution to equation (3.7), then this set of numbers is also a solution to inequality (3.7).

There is $n-k$ equations of the form (3.7) with k unknowns, the right-hand side of which is less than n (but is not less than k, as there is no solution in this case). Below is the list of such equations (in ascending order with respect to the right-hand sides):

$$\begin{gathered}
x_1+x_2+x_3+\ldots+x_k=k, \\
x_1+x_2+x_3+\ldots+x_k=k+1, \\
x_1+x_2+x_3+\ldots+x_k=k+2, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_1+x_2+x_3+\ldots+x_k=n-2, \\
x_1+x_2+x_3+\ldots+x_k=n-1.
\end{gathered} \tag{3.9}$$

The solutions of every equation form the part of solutions to inequality (3.7), and there are no other solutions to this inequality. Another important observation is that there are no common solutions to any two of the above equations (as the left-hand sides are equal and the right-hand sides are not). From the above, we conclude: having counted all solutions to equations (3.9), we obtain the number of solutions to inequality (3.7).

We emphasize that every time when we mention the solutions to equations (3.9) or inequality (3.7) we mean the natural solutions.

According to (3.32), (3.8) has C_{s-1}^{k-1} natural solutions. Therefore, inequality (3.7) has

$$C_{k-1}^{k-1}+C_{k}^{k-1}+C_{k+1}^{k-1}+\ldots+C_{n-3}^{k-1}+C_{n-2}^{k-1}$$

such solutions. Is there a way to reduce this sum and express the answer in more compact form? Yes, there is. We can achieve this using the most important recurrence relation for the numbers C_s^t:

$$C_s^t = C_{s-1}^t + C_{s-1}^{t-1}.$$

This formula holds for any natural s greater than 1, and any t from the interval from 1 to $s-1$. We have considered this formula in Exercise 9. C_s^t is the amount of all t-element subsets of an s-element set, C_{s-1}^t is the amount of those subsets that do not include some element of this set, and C_{s-1}^{t-1} is the amount of subsets including this element.

In order to reduce the above sum, we repeatedly apply this formula having replaced the symbol C_{k-1}^{k-1} (which is the initial summand) by the equivalent amount C_k^k beforehand. Then we apply the major recurrence relation step by step. Every time the number of summands decreases by 1 until there remains only one. We present several initial and ending elements of this chain of transformations:

$$\begin{array}{l}
C_k^k + C_k^{k-1} = C_{k+1}^k,\\
C_{k+1}^k + C_{k+1}^{k-1} = C_{k+2}^k,\\
C_{k+2}^k + C_{k+2}^{k-1} = C_{k+3}^k,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
C_{n-3}^k + C_{n-3}^{k-1} = C_{n-2}^k,\\
C_{n-2}^k + C_{n-2}^{k-1} = C_{n-1}^k.
\end{array}$$

Instead of a somewhat unpleasant sum which in addition depends on n and k, we get a concise formula C_{n-1}^k. It can inspire to development of a fundamentally different less straightforward but more effective solution. The final formula can even advise the direction of the investigation. It appears that the number C_{n-1}^k, which provides the answer to the question about the number of natural solutions to the inequality (3.7), defines the number of natural solutions to the equation

$$x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1} = n, \tag{3.10}$$

which contains one more unknown variable than inequality (3.7) (this is the result of 3.32). This coincidence is an explicit hint: it is worth a try to establish a bijection between the natural solutions to inequality (3.7) and natural solutions to equation (3.10).

Second Approach consists in search of this bijection. The rule defining the correspondence between the solutions to equation (3.10) and the solutions to inequality (3.7) is simple and obvious. There is no need to dig deep. If $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k, \gamma_{k+1})$ is a solution to the equation, then $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k)$ is a solution to the inequality. Why? Because $\gamma_{k+1} > 0$, as we consider natural solutions only. The equality

$$\gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_k + \gamma_{k+1} = n$$

yields that

$$\gamma_1 + \gamma_2 + \gamma_3 + \ldots + \gamma_k = n - \gamma_{k+1} < n.$$

Conversely, if $\beta_1, \beta_2, \beta_3, ..., \beta_k, \beta_k + 1)$ is a solution to inequality (3.7), that is

$$\beta_1 + \beta_2 + ... + \beta_k < n,$$

then the number

$$\beta_{k+1} = n - (\beta_1 + \beta_2 + \beta_3 + ... + \beta_k)$$

is positive and $(\beta_1, \beta_2, \beta_3, ..., \beta_k, \beta_{k+1})$ is a natural solution to equation (3.10).

Therefore, if we shorten a solution to the equation removing its last component, then we get a solution to the inequality; conversely, a natural component can be attached to any solution of the inequality to turn it into a solution to the equation. This establishes a bijection between two sets of solutions, which in particular evidences that the amounts of solutions are the same.

Problem 3.36. *How many natural solutions are there to the compound inequality*

$$m \leq x_1 + x_2 + x_3 + ... + x_k < n$$

with k unknowns $x_1, x_2, x_3, ..., x_k$*?*

Answer. $C_{n-1}^k - C_{m-1}^k$.

Problem 3.37. *How many integer non-negative solutions are there to the compound inequality*

$$x_1 + x_2 + x_3 + ... + x_k < n \tag{3.11}$$

with k unknowns $x_1, x_2, x_3, ..., x_k$*?*

Answer. C_{n+k-1}^k.

Sketch of Solution. Integer non-negative solutions to inequality (3.11) are in bijective correspondence with those integer non-negative solutions to the equality

$$x_1 + x_2 + x_3 + ... + x_k + x_{k+1} = n, \tag{3.12}$$

that have positive last component. The law of correspondence: a solution $(\gamma_1, \gamma_2, \gamma_3, ..., \gamma_k)$ to inequality 3.11 can be extended to a solution for equation (3.12) by attaching the component γ_{k+1}, defined by the equality

$$\gamma_{k+1} = n - (\gamma_1 + \gamma_2 + \gamma_3 + ... + \gamma_k).$$

On the other hand, those integer non-negative solutions to equation (3.12) that have positive last component, are in bijective correspondence with integer non-negative solutions to the equation

$$x_1 + x_2 + x_3 + ... + x_k + x_{k+1} = n - 1. \tag{3.13}$$

The law of correspondence: a solution

$$(t_1, t_2, t_3, ..., t_k, t_{k+1})$$

to equation (3.12) with positive component t_{k+1} matches to the solution

$$(t_1, t_2, t_3, ..., t_k, t_{k+1} - 1)$$

to equation (3.13).

Problem 3.38. *How many integer non-negative solutions to the inequality*

$$x_1 + x_2 + x_3 + \ldots + x_{k-1} + x_k < n$$

have positive first component?

Answer. C^k_{n+k-2}.

Hint. Such solutions are in bijective correspondence with all integer non-negative solutions to the inequality

$$x_1 + x_2 + x_3 + \ldots + x_{k-1} + x_k < n-1. \tag{3.14}$$

The law of correspondence: a solution

$$(1+\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{k-1}, \gamma_k)$$

to the original inequality matches to the solution

$$(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{k-1}, \gamma_k)$$

to inequality (3.14). Then one needs to use the previous Exercise.

Problem 3.39. *How many integer solutions with non-zero components are there to the equation*

$$|x_1| + |x_2| + |x_3| + \ldots + |x_{k-1}| + |x_k| = n$$

with k unknowns $x_1, x_2, x_3, \ldots, x_{k-1}, x_k$ (n is given natural number)?

Answer. $2^k C^{k-1}_{n-1}$ *if* $n \geq k$; 0 *if* $n < k$.

Clarification. The equation

$$x_1 + x_2 + x_3 + \ldots + x_{k-1} + x_k = n$$

has C^{k-1}_{n-1} solutions with natural components. From every of these solutions one can get 2^k different solutions to the equation in question arbitrarily choosing "+" or "–" signs in front of each component.

Problem 3.40. *How many integer solutions with non-zero components are there to the inequality*

$$|x_1| + |x_2| + |x_3| + \ldots + |x_{k-1}| + |x_k| < n?$$

Answer. $2^k \cdot C^k_{n-1}$.

Problem 3.41. *How many natural solutions $(x; y; z; t)$ are there to the system of equations*

$$\begin{cases} x + y + z = k, \\ y + z + t = k \end{cases}$$

(k is a given natural number)?

Answer. C^2_{k-1}.

Problem 3.42. *How many natural solutions* $(x; y; z; u; t)$ *are there to the system of equations*

$$\begin{cases} x+y+z+u=k, \\ y+z+u+v=m \end{cases}$$

(k and m are fixed natural numbers)?

Answer. C_{s-1}^3, *where s is the least of two numbers k and m. Clearly, there is no natural solution to the system if* $s<4$.

Problem 3.43. *How many natural solutions* $(x; y; z; u; v)$ *are there to the system of equations*

$$\begin{cases} x+y+z=k, \\ z+u+v=k \end{cases}$$

(k is a given natural number)?

Answer. $1^2+2^2+3^2+...+(k-2)^2 = \frac{(k-2)(k-1)(2k-3)}{6}$.

Hint. Find the amount of those solutions that have the fixed component $z\,(z = 1, 2, 3, ..., k-2)$.

Problem 3.44. *How many natural solutions* $(x; y; z; t; u; v)$ *are there to the system of equations*

$$\begin{cases} x+y+z+t=n+3, \\ z+t+u+v=n+3 \end{cases}$$

(n is a given natural number)?

Answer. $\frac{1}{2}C_{n+2}^3 \cdot C_{n+1}^1$.

Solution. Not very original, yet reliable counting technique is to split all possible solutions into groups depending on the sum of values of variables z and t. The process can be reduced to filling in the following table 3.1.

Table 3.1. Sum of values z and t.

Sum of z and t	Number of options for			Number of solutions to the system
	z and t	x and y	u and v	
2	1	n	n	$1\cdot n^2$
3	2	$n-1$	$n-1$	$2\cdot(n-1)^2$
4	3	$n-2$	$n-2$	$3\cdot(n-2)^2$
5	4	$n-3$	$n-3$	$4\cdot(n-3)^2$
....	....	....	....	....
$n-1$	$n-2$	3	3	$(n-2)\cdot 3^2$
n	$n-1$	2	2	$(n-1)\cdot 2^2$
$n+1$	n	1	1	$n\cdot 1^2$

The answer to the question of the problem

$$S_n = 1 \cdot n^2 + 2(n-1)^2 + 3(n-2)^2 + \ldots + (n-1) \cdot 2^2 + n \cdot 1^2$$

is absolutely acceptable, but one might want to reduce the sum and express the result with a formula, the "length" of which does not depend on n. For example, this can be achieved as follows.

Note that our sum is the sum of squares of initial natural numbers:

$$S_n = 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots + (1^2 + 2^2 + 3^2 + \cdots + k^2) + \\ \cdots + (1^2 + 2^2 + 3^2 + \cdots + n^2).$$

Reducing all these sums of squares

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k; \; k = 1, 2, \ldots, n,$$

we express S_n as the sum of three sums:

$$S_n^{(1)} = \frac{1}{3}(1^3 + 2^3 + 3^3 + \cdots + n^3) = \frac{n^2(n+1)^2}{12};$$

$$S_n^{(2)} = \tfrac{1}{2}(1^2 + 2^2 + 3^2 + \cdots + n^2) = \tfrac{n(n+1)(2n+1)}{12};$$

$$S_n^{(3)} = \tfrac{1}{6}(1 + 2 + 3 + \cdots + n) = \tfrac{n(n+1)}{12}.$$

Finally, we get the following formula for the wanted amount S_n:

$$S_n = S_n^{(1)} + S_n^{(2)} + S_n^{(3)} = \frac{n(n+1)^2(n+2)}{12}.$$

It is eye-catching that the latter formula can be expressed with the numbers C_s^k. Moreover, there are two essentially different ways to make this:

$$S_n = \frac{1}{3} C_{n+1}^2 \cdot C_{n+2}^2 \tag{3.15}$$

or

$$S_n = \frac{1}{2} C_{n+2}^3 \cdot C_{n+1}^1. \tag{3.16}$$

These elegant formulas suggest that there could exist a technique of calculation of the number of solutions to a system of equations that can lead us straight to the answers in the form (3.15) or (3.16), skipping all summations. Try to find such a method.

Remark. The formulas

$$\sigma_n^{(3)} = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$$

and

$$\sigma_n^{(2)} = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

can be proved (but not derived!), as follows.

1 The value of $\sigma_n^{(3)}$ obeys the recurrence relation

$$\sigma_n^{(3)} = \sigma_{n-1}^{(3)} + n^3$$

with the initial condition $\sigma_1^{(3)} = 1$. Hence in order to check the direct formula

$$\sigma_n^{(3)} = \frac{n^2(n+1)^2}{4},$$

it suffices to verify that it satisfies the recursive formula and the initial condition. We have:

$$\sigma_{n-1}^{(3)} + n^3 = \frac{(n-1)^2 n^2}{4} + n^3 = \frac{n^2}{4} \cdot ((n-1)^2 + 4n) = \frac{n^2(n+1)^2}{4} = \sigma_n^{(3)};$$

$$\sigma_1^{(3)} = \frac{1^2 \cdot 2^2}{4} = 1.$$

The formula is proved.

2. The value of $\sigma_n^{(2)}$ obeys the recurrence relation

$$\sigma_n^{(2)} = \sigma_{n-1}^{(2)} + n^2$$

with the initial condition $\sigma_1^{(2)} = 1$. We check if the above equalities are satisfied for the hypothetical direct formula for $\sigma_n^{(2)}$. We have:

1. $\sigma_{n-1}^{(2)} + n^2 = \frac{(n-1)2(2n-1)}{6} + n^2 = n \cdot \frac{(n-1)(2n-1)+6n}{6} = \frac{n(n+1)(2n+1)}{6} = \sigma_n^{(2)};$
2. $\sigma_1^{(2)} = \frac{1 \cdot 2 \cdot 3}{6} = 1.$

Thus the direct formula for $\sigma_n^{(2)}$ is proved.

Problem 3.45. *1. How many integer solutions does the equation*

$$|x| + |y| + |z| = n \tag{3.17}$$

have (n is a given natural number)?

2. Which geometric interpretation can be given to equation (3.17)*, and what is the geometric meaning of its integer solutions?*

Answer. *1)* $4n^2 + 2$.

Solution. 1. It follows from the result of Exercise 39 that the equation in question has $2^3 \cdot C_{n-1}^2$ solutions with non-zero components. There are $3 \cdot 2^2 \cdot C_{n-1}^1$ and $3 \cdot 2 \cdot C_{n-1}^0$ solutions with one and two components being zero respectively. Therefore, there are $2^3 \cdot C_{n-1}^2 + 3 \cdot 2^2 \cdot C_{n-1}^1 + 3 \cdot 2 \cdot C_{n-1}^0 = 4n^2 + 2$ integer solutions in total.

2. Equation (3.17) does not change upon the change of signs of variables. This evidences that the surface that is its geometric analog is symmetrical with respect to all three coordinate planes (and with respect to all three coordinate axes and the point of origin). In order to complete the picture of this surface, it suffices to find the shape of its part, which

lays in the first octant and then exploit the observed symmetries. In the first octant, all coordinates of all points are positive or zero. Thus equation (3.17) has a more simple form in the first octant:

$$x+y+z=n.$$

This is a linear equation that has a plane in space corresponding to it. In the first octant, there is a “minor” part of this plane, which is the equilateral triangle with vertices in the points $(n;0;0)$, $(0;n;0)$ and $(0;0;n)$. This triangle and its reflections with the reflections of the latter concerning the coordinate planes form the wanted surface, which is the geometric image of equation (3.17). This is the surface of a regular octahedron (a polyhedron with eight faces). Its six vertices lay on the coordinate axes at a distance of n from the point of origin, and its twelve edges lay in the coordinate planes, with four of them (forming a square) belonging to each. Finally, each of its eight faces belongs to a different coordinate octant. The integer solutions to equation (3.17) are the points with integer coordinates laying on the surface of the octahedron. The points with non-zero coordinates belong to the interior of its faces, and those with one or two zero coordinates lay in the interior of its edges and its vertices respectively.

Problem 3.46. *How many three-digit numbers consist of three different digits with the second digit (the digit denoting tens) being the greatest of the three?*
Answer. $2\cdot C_9^3+C_9^2=204$.

Sketch of Solution. One could count the numbers with and without zero among its digits separately. The result will be the answer.

Alternatively, another technique can be applied. The second digit could be any digit except 0 and 1. Let us count the amount of numbers that have the second digit equal to p $(p=2,3,4,\dots,9)$. The first digit (the digit denoting hundreds) could be $1,2,3,\dots,p-2$ or $p-1$, and the third could be any digit from 0 to $p-1$, except for the one chosen for the first position. Therefore, by the combinatorial rule of product there are $(p-1)^2$ numbers having p as their second digit, and overall there are

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2$$

numbers in question

Problem 3.47. *How many three-digit numbers are there, where the second digit is greater than the other two?*
Answer. *240.*

Hint. In the previous problem, we have dealt with part of these numbers, namely those which have different digits in places of hundreds and ones. There are 204 such numbers. Now, it suffices to count the numbers having the same digits in the first and third places.

Problem 3.48. *1. How many three-digit numbers are there, where each consecutive digit is greater than or equal to the previous one?*

2. How many four-digit numbers are there, where each consecutive digit is greater than or equal to the previous one?
Answer.

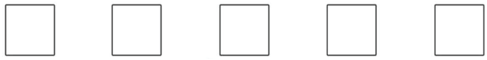

Figure 3.2. Five-digit numbers (a).

1. C_{11}^{3}*;*

2. C_{12}^{4}*.*

Solution.

First Approach. The number in the question can not contain zero. We split all numbers into three groups depending on the amount of different digits composing them: one, two, or three. The first group contains C_9^3 numbers, the second has $2 \cdot C_9^2$, and the third has C_9^1 numbers. So there are $C_9^3 + 2C_9^2 + C_9^1 = 165$ numbers in total.

The above counting method is absolutely transparent and natural but its application to numbers with more digits is concerned with essential technical difficulties. However, it is rewarding to overcome these difficulties and get a nice formula as a result.

Let us apply the above technique to count five-digit numbers, where each next digit is greater than or equal to the previous one. There is no such number with digit zero in it. Other digits have "equal rights" in the construction of the numbers of interest. The only limitation is that there could be no more than five different digits in each number (as there are only five positions in them). First, we split the wanted numbers into five groups and determine the amount of numbers in each group separately:

1. the numbers composed of five different digits;
2. the numbers composed of four different digits
3. the numbers composed of three different digits;
4. the numbers composed of two different digits;
5. the numbers constructed with only one digit.

To present the key idea in the most straightforward way, it is convenient to begin with the third group instead of the first. So let us answer the question: how many numbers of interest consist of exactly three different digits? In fact, it splits into two separate questions:

1. How many ways to choose digits for our numbers are there?
2. How many different numbers can be composed with exact three different digits?

The product of answers to the above questions is the amount of numbers in the third group. Question (1) is of a standard type. It asks about the number of ways to choose three objects out of available nine (digits from 1 to 9). The answer is as usual: C_9^3. The idea behind the brief and simple answer to question (2) is a little less obvious. Let us choose three arbitrary digits, say, $1, 2$, and 3. A five-digit number has 5 positions, which can be illustrated with a row of squares:

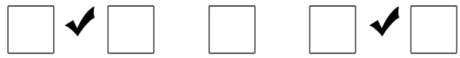

Figure 3.3. Five-digit numbers (b).

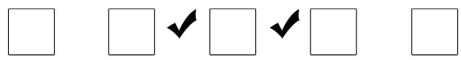

Figure 3.4. Five-digit numbers (c).

There are 4 intervals between these squares. Every interval separates two neighboring squares. We choose any two of these intervals (say, the first and the fourth) and mark them with ticks.

The resulting configuration defines the exact number of interest 1 2 2 2 3 (we put 1 in all squares standing to the left from the first tick, 2 in all squares between the ticks and 3 in the squares to the right from the second tick). Conversely, any number of interest consisting of the digits 1, 2 and 3 has some configuration of five squares and two ticks between them corresponding to it. For example, the number 11233 corresponds to the configuration

In other words, there is a bijection between the objects of two types (the numbers and configurations), which means that the amounts of objects of each type are the same. There are C_4^2 ways to choose two intervals out of 4. Therefore, the amount of five-digit numbers is the same: C_4^2.

Thus, we have derived the answer to the question (3) about the amount of numbers in the third group: $C_9^3 \cdot C_4^2$.

Similarly, we can count the numbers in other groups.

In total, the amount of five-digit numbers that consist of non-decreasing digits from left to right is the following:

$$C_9^5 \cdot C_4^4 + C_9^4 \cdot C_4^3 + C_9^3 \cdot C_4^2 + C_9^2 \cdot C_4^1 + C_9^1 \cdot C_4^0.$$

Now, turn back to the three-digit numbers from the first question of the problem. They can be counted in a different way. An alternative approach is based on the existence of a bijection between the wanted numbers and subsets of some set. The fact of the existence of such a bijection is far from being obvious, and the discovery of this fact heavily impresses.

Second Approach. Let $\overline{abc}$ be a number, the digits of which satisfy the compound inequality

$$a \le b \le c. \tag{3.18}$$

Additionally, assume $a \ge 1$, as the first digit of a three-digit number can not be zero. Let us match with the number $\overline{abc}$ the triplet $(a; b+1; c+2)$. What can we say about the components of this triplet, the numbers a, $b+1$ and $c+2$? First, they are different numbers. This results from inequality (3.18): $a < b+1$, as $a \le b$, and $b+1 < c+2$, as $b \le c$. Moreover, none of them is less than 1 or greater than 11. This is true, because $a \ge 1$ and $c \le 9$ (as c is a digit).

$$\overline{abc} \to (a; b+1; c+2) \tag{3.19}$$

Hence, the correspondence matches every three-digit number that has its every consecutive digit greater than the previous one (moving from left to right), with the triplet of different natural numbers less than 11. Now, we have to make sure that this correspondence is a bijection between all three-digit numbers of interest and all possible triplets of different natural numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

Let $(p; q; r)$ be arbitrary triplet of natural numbers, for which the inequalities

$$1 \le p < q < r \le 11$$

hold. Then for the numbers $a = p,\ b = q - 1$ and $c = r - 2$ the inequalities

$$1 \le a \le b \le c \le 9,$$

hold, hence $\overline{abc}$ is a three-digit number satisfying the stated condition.

The last observation evidence that the correspondence (3.19) is a bijection. Therefore, the amount of wanted three-digit numbers is the same as the amount of three-element subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. The latter amount is known and equals C_{11}^3. This is the answer to the first question of the problem.

The approach described above enables one to give an immediate answer to the question about the amount of numbers formed by non-decreasing sequences of digits of any (fixed) length. For the case of n-digit numbers, the answer is C_{n+8}^n (Why?). In particular, for $n = 5$ we get C_{13}^5. Comparing this result with the outcome of the first approach, we conclude that the following inequality holds:

$$C_9^5 \cdot C_4^4 + C_9^4 \cdot C_4^3 + C_9^3 \cdot C_4^2 + C_9^2 \cdot C_4^1 + C_9^1 \cdot C_4^0 = C_{13}^5.$$

It is hard just to imagine a clearly arithmetical approach that can lead to that type of equality. Combinatorial problems are the source of many interesting equalities for numbers C_s^t. These equalities are usually obtained when a problem is solved in two fundamentally different ways. So the problem serves as a source of equality and as its proof at the same time.

Problem 3.49. *Formulate the previous exercise in the general case (for n-digit numbers) Apply both approaches suggested in the solution to the previous problem to derive equality for the numbers C_s^t, which is the generalization of the equality for $n = 5$ from the previous exercise.*

Answer. $C_9^n \cdot C_{n-1}^n + C_9^{n-1} \cdot C_{n-1}^{n-2} + \ldots + C_9^1 \cdot C_{n-1}^0 = C_{n+8}^n$.

The left-hand side of this equality requires additional clarification. We already know that the symbol C_n^k has combinatorial sense only if n is natural, and k is an integer from the interval $[0, n]$. When this is the case, C_n^k is the amount of k-element subsets of an n-element set (in other words, how many ways are there to choose k elements out of available n). There can be two agreements concerning the symbols C_n^k for natural k, which is greater than n: they can be declared senseless or equal to zero. There are no shortcomings in both options. The calculations and formulas often get easier when C_n^k are deemed well-defined in the case $k > n$. This is the way to understand the left-hand side of our equality. If $n > 9$, then there are symbols equal to zero in it.

Problem 3.50. *1. How many three-digit numbers have their digits standing in non-ascending order from left to right (the digit denoting tens is less than or equal to the digit of hundreds and the digit denoting ones is less than or equal to the digit denoting tens)?*

2. How many four-digit numbers have their digits standing in non-ascending order?

3. How many n-digit numbers have their digits standing in non-ascending order?

Answer.

1. $C_{12}^3 - 1$;

2. $C_{13}^4 - 1$;

3. $C_{n+9}^n - 1$.

Hint. Begin with thorough investigation of the solutions of two previous problems.

Problem 3.51. *John has 6 squares of paper. Digit 1 is printed on two of them, digit 2 is printed on two others, and digit 3 is on the other two. How many different six-digit numbers can John create by placing the squares next to each other? What is the sum of these numbers?*

Answer. 90; 19999980.

Solution. In order to construct six-digit numbers with available cards, John has to decide:

1. which will be the positions for the digit 1 (there are C_6^2 ways to make it);
2. which two of the remaining four positions will the digit 2 occupy (there are C_4^2 ways to choose them).

The last two places will be filled with the digit 3 without an alternative.

Therefore, overall there are $C_6^2 \cdot C_4^2 = 90$ numbers.

Let us choose any position of these ninety numbers and ask ourselves: how many times will each of the digits 1, 2, and 3 appear in this position? Obviously, none of the digits has an advantage over the others, as they take part in the construction of ninety numbers on equal rights. Therefore, each digit appears in each position an equal amount of times – 30. So the sum of all digits in all positions is $(1+2+3)\cdot 30 = 180$, which means that the sum of all numbers is

$$180\cdot(1+10+10^2+10^3+10^4+10^5) = 180\cdot 111111 = 19999980.$$

Problem 3.52. *How many three-digit numbers consist of one or two different digits?*

Answer. 252.

Solution. First Approach. There are 9 numbers that consist of one-digit.

There are C_9^2 ways to choose two digits, none of which is zero. These two digits appear in six numbers (under the condition that both digits should be used), e.g.:

112, 121, 211, 221, 212, 122.

Two digits, one of which is zero, can construct 3 numbers (again, both digits should participate), e.g.:

100, 110, 101.

So in total, there are $9+6\cdot C_9^2+3\cdot 9=252$ numbers.

Second Approach. Overall, there are 900 three-digit numbers. $9\cdot 9\cdot 8=648$ of them consist of three different digits, while other $(900-648=252)$ are constructed with one or two digits.

Problem 3.53. *How many four-digit numbers can be written with one or two digits?*

Answer. 576.

One of the Possible Solutions. There are C_9^2 ways to choose two digits none of which is zero. Two digits can construct 2^4-2 four-digit numbers (with both digits appearing in every number).

There are 9 ways to choose two digits, one of which is zero. There exist 2^3-1 numbers created with these digits and 9 numbers composed of only one digit. Therefore, we have

$$C_9^2\cdot(2^4-2)+9\cdot(2^3-1)+9=576$$

numbers of interest in total.

Problem 3.54. *How many six-digit numbers have their digits ordered as follows: the first three digits are in descending order and the last three are in ascending, where the third and fourth digits are ordered arbitrarily (e.g. 860245, 321169, 974037)?*

Answer. $(C_{10}^3)^2$.

Problem 3.55. *How many eight-digit numbers have their digits ordered as follows: the first four digits are in descending order and the last four are in ascending, and the fourth and fifth digits are ordered arbitrarily?*

Answer. $C_9^4\cdot C_{10}^4$.

Problem 3.56. *Alex occasionally travels by bus. Several times in a row, Alex has been buying bus tickets with an interesting feature: the first three digits are standing in descending order and the last three are ordered ascendingly. In addition, two middle digits (the third and fourth) are equal. Being used to find mathematical problems in the surrounding world, Alex wonders how many such tickets exist. Find the answer to this question.*

Answer. $(C_2^2)^2+(C_3^2)^2+(C_4^2)^2+(C_5^2)^2+(C_6^2)^2+(C_7^2)^2+(C_8^2)^2+(C_9^2)^2$.

Problem 3.57. *If a number is expressed in binary numeral system than only two digits (0 and 1) are needed instead of ten. For example, 1011011 denotes the number* $1+1\cdot 2+0\cdot 2^2+1\cdot 2^3+1\cdot 2^4+0\cdot 2^5+1\cdot 2^6=91$.

Assume all numbers are expressed in binary numeral system.

1. *How many n-digit numbers are there?*

2. *How many of these n-digit numbers have the sum of their digits equal to k?*

Answer.

1. 2^{n-1};
2. C_{n-1}^{k-1}.

Problem 3.58. *Every batch of local bus tickets contains tickets with numbers ranging from 000 to 999.*

1. *How many tickets are there in one batch?*
2. *Let k be a number from 0 to 9. How many tickets in one batch have the sum of their digits equal to k? Construct a table with two rows and 10 columns, where the first row contains possible values of k ordered ascendingly from 0 to 9, and in the second row there are the amounts of tickets that have a corresponding sum of digits.*
3. *Establish a bijection between the numbers of tickets that have the sums of digits in their numbers equal to k, and the numbers of those tickets, the sums of digits of numbers of which equal to* $27-k$. *Basing on the previous question, find the amount of tickets that have the sums of digits of their numbers equal to s, where* $18 \le s \le 27$.
4. *Prove that the equation*
$$x+y+z=10+c,$$
where $0 \le c \le 9$, *has three times more non-negative solutions, which have one of the values of unknowns greater than 9, than the equation*
$$x+y+z=c$$
has integer non-negative solutions.
5. *Basing on the result of the previous paragraph, determine the amount of those tickets in one batch that have the sums of digits of their numbers equal to* $10+c$ $(0 \le c \le 9)$.
6. *For the sums of digits equal to 18 and 19, the amount of numbers of tickets can be found in two different ways: the one presented in paragraph 3) and the method from the previous paragraph. Perform calculations and compare the results.*
7. *From the results obtained in paragraph 3), it follows that the amounts of numbers of tickets with the sums of their digits 10 and 17, 11 and 16, 12 and 15, finally, 13 and 14 should be the same. These quantities are derived in paragraph 5). Ensure that the amounts of numbers is really the same for each pair of sums of digits.*

Solution.

2. If k is less than or equal to 9, then the amount of numbers of tickets the sums of digits of which are equal to k, is the same as the amount of integer non-negative solutions to the equation
$$x+y+z=k$$
(there is an obvious bijection between the solutions to these equations and the wanted three-digit numbers). It remains to recall that this equation has C_{k+2}^{2} integer non-negative solutions. This fact is one of the results obtained when solving problem 3.33. See table 3.2.

Table 3.2. Integer non-negative solutions

k	0	1	2	3	4	5	6	7	8	9
C_{k+2}^2	1	3	6	10	15	21	28	36	45	55

For example, we list all those three-digit numbers that have the sums of their digits equal to $3 : 300, 030, 003, 210, 201, 120, 021, 102, 012, 111.$

3. Let abc be a three-digit number with the sum of its digits equal to k (a, b and c are digits and $a+b+c=k$). Then $u = 9-a$, $v = 9-b$ and $t = 9-c$ are also digits and $u+v+t = 27-(a+b+c) = 27-k$. Conversely, if uvt is a three-digit number with the sum of digits $27-k$, then abc, where $a = 9-u$, $b = 9-v$, $c = 9-t$, is a three-digit number with the sum of its digits equal to $a+b+c = 27-(u+v=t) = k$. Thus, for every number, which has the sum of its digits equal to k, there is a three-digit number corresponding to it, which is composed of the digits that are the results of subtraction of the digits of the original number from 9. The latter number has the sum of its digits equal to $27-k$. The correspondence between the numbers is bijective. For instance, the list of corresponding numbers with the sums of digits equal to 3 and 24 is presented below:

$$\begin{array}{ll}
300 \leftrightarrow 699 & 120 \leftrightarrow 879 \\
030 \leftrightarrow 969 & 021 \leftrightarrow 978 \\
003 \leftrightarrow 996 & 102 \leftrightarrow 897 \\
210 \leftrightarrow 789 & 012 \leftrightarrow 987 \\
201 \leftrightarrow 798 & 111 \leftrightarrow 888
\end{array}$$

The above considerations evidence that the amount of numbers with the sums of digits equal to k is the same as the amount of numbers with the sums of digits equal to $27-k$.

The table 3.2, containing information about the amount of numbers with the sums of digits from 0 to 9, can now be significantly supplemented by the following table 3.3.

Table 3.3. Sum of digits

k Sum of digits	0 27	1 26	2 25	3 24	4 23	5 22	6 21	7 20	8 19	9 18
Amount of numbers	1	3	6	10	15	21	28	36	45	55

It is straightforward to find the formula for the amount of numbers with the sums of digits equal to s, where s comes from the interval from 18 to 27. Indeed, the amount of such numbers is the same as the amount of numbers that have the sum of digits equal to $27-s$. The latter quantity equals to

$$C_{29-s}^2,$$

as it has been discovered in the previous paragraph (because $0 \leq 27-s \leq 9$ if $18 \leq s \leq 27$).

4. The problem is about the comparison of integer non-negative solutions to the equation

$$u+v+t=c, \tag{3.20}$$

where c is a number from the interval from 0 to 9, with that integer non-negative solutions to the equation

$$x+y+z=10+c, \tag{3.21}$$

one of the components (one of the values of unknowns) of which is 10 or greater. Besides, no solution can have two components greater than 9, because $c \le 9$.

Let $(p;q;r)$ be an integer non-negative solution to equation (3.20). Then $(10+p;q;r)$, $(p;\,10+q;r)$ and $(p;\,1;\,10+r)$ are three different solutions to equation (3.21), all of which have a component greater than 9. There could be no repetitions during this process of replication of the solutions to equation (3.21), based on the solutions to equation (3.20). Really, if $(p_1;q_1;r_1)$ is a solution to equation (3.20), different from $(p;q;r)$, then the triplet $(10+p_1;q_1;r_1)$ can not be the same as the triplet $(10+p;q;r)$. The same concerns to two other pairs of triplets. This means that basing on the solutions to (3.20), one can create three times as many solutions to equation (3.21). The latter solutions will have a component greater than 9. There are no other solutions of this type to equation (3.21): if we assume that $(x_0;y_0;z_0)$ is such solution, and, say, $x_0 > 9$, then $(x_0-10;y_0;z_0)$ is a solution to equation (3.20) with non-negative components.

Let us illustrate the above with the list of corresponding solutions to equations (3.20) and (3.21) for some given value of c, say, $c=3$. The left column of the following table 3.4 contains the list of all solutions to the equation $u+v=t=3$. For each such solution, there are three corresponding solutions to the equation $x+y+z=13$ standing in the same row in the right column.

Table 3.4. Number of integer non-negative solutions to equation $u+v=t=3$

Solutions to the equation $u+v=t=3$	Those solutions to the equation $x+y+z=13$ that have a component greater than 9
(3; 0; 0)	(13; 0; 0), (3; 10; 0), (3; 0; 10)
(0; 3; 0)	(10; 3; 0), (0; 13; 0), (0; 3; 10)
(0; 0; 3)	(10; 0; 3), (0; 10; 3), (0; 0; 13)
(2; 1; 0)	(12; 1; 0), (2; 11; 0), (2; 1; 10)
(2; 0; 1)	(12; 0; 1), (2; 10; 1), (2; 0; 11)
(1; 2; 0)	(11; 2; 0), (1; 12; 0), (1; 2; 10)
(0; 2; 1)	(10; 2; 1), (0; 12; 1), (0; 2; 11)
(1; 0; 2)	(11; 0; 2), (1; 10; 2), (1; 0; 12)
(0; 1; 2)	(10; 1; 2), (0; 11; 2), (0; 1; 12)
(1; 1; 1)	(11; 1; 1), (1; 11; 1), (1; 1; 11)

5. The sought amount is the difference between the number of integer non-negative solutions to equation (3.21) and the amount of those solutions to equation (3.21) that have a component greater than 9. According to the previous paragraph, the latter number is

three times greater than the amount of integer non-negative solutions to equation (3.21). Therefore, we have: if $0 \leq c \leq 9$, then there are

$$C_{12+c}^{2} - 3C_{2+c}^{2}$$

numbers of tickets, which have the sums of their digits equal to $10 + c$. For example, there are

$$C_{15}^{2} - 3C_{5}^{2} = 105 - 30 = 75$$

numbers of tickets that have the sums of their digits equal to 13. We provide the complete table of these numbers for $c = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ (see Table IV).

6. The last two rows of the table 3.5 contains numbers 55 and 45, which represent the amounts of numbers with the sums of digits equal to 18 and 19 respectively. The same numbers are in appropriate positions in the table 3.3.

Table 3.5. The number of tickets

C	The number of tickets in which the number of digits is $C + 10$
0	$C_{12}^{2} - 3C_{2}^{2} = 63$
1	$C_{13}^{2} - 3C_{3}^{2} = 69$
2	$C_{14}^{2} - 3C_{4}^{2} = 73$
3	$C_{15}^{2} - 3C_{5}^{2} = 75$
4	$C_{16}^{2} - 3C_{6}^{2} = 75$
5	$C_{17}^{2} - 3C_{7}^{2} = 73$
6	$C_{18}^{2} - 3C_{8}^{2} = 69$
7	$C_{19}^{2} - 3C_{9}^{2} = 63$
8	$C_{20}^{2} - 3C_{10}^{2} = 55$
9	$C_{21}^{2} - 3C_{11}^{2} = 45$

7. All necessary information is given in the table 3.5.

Besides, the double of the sum of numbers from the second row of the table 3.3. along with the sum of numbers from the second column of the table 3.5, except for the last two (as they have already been accounted for the table 3.3), must equal to the total amount of numbers of tickets from 000 to 999, which is 1000. The following relation holds true:

$$2 \cdot (1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 63 + 69 + 73 + 75) = 1000.$$

This is a good sign, which suggests that there is no mistake in our calculations.

Problem 3.59. *Bus tickets are enumerated with six digits: from 000000 to 99999. A ticket is deemed to be lucky if the sum of the first three digits is equal to the sum of the last three. How many lucky tickets are there?*

<u>*Answer.*</u> 55252.

Solution. Denote by $s(k)$ the amount of those combinations of three digits, the sum of which equals k. For example, $s(1) = 3$, as there are only three different groups of three digits that sum up to 1, namely: $100, 010$ and 001. It is straightforward to calculate the amount of lucky tickets that have the sum of their first and last three digits equal to 1. In order for this to happen, left and right groups of digits should be 100, or 010, or 001. There are three options for both groups, which can be combined arbitrarily. Therefore, there are $3 \cdot 3 = 9$ lucky tickets of this type. Similarly, there are $s(k) \cdot s(k) = (s(k))^2$ numbers both groups of digits of which sum up to k. The numbers $s(k)$ have been determined in the previous problem. In particular, it has been proved that $s(k) = s(27 - k)$. Taking this into account we conclude that there are

$$\begin{aligned}
&2 \cdot (s^2(0) + s^2(1) + s^2(2) + s^2(3) + s^2(4) + s^2(5) + s^2(6) + s^2(7) + \\
&+ s^2(8) + s^2(9) + s^2(10) + s^2(11) + s^2(12) + s^2(13)) = \\
&= 2 \cdot (1^2 + 3^2 + 6^2 + 10^2 + 15^2 + 21^2 + 28^2 + 36^2 + \\
&+ 45^2 + 55^2 + 63^2 + 69^2 + 73^2 + 75^2) = 55252
\end{aligned}$$

lucky tickets in total. Thus, around $5,5\%$ of all tickets are lucky (and overall there are 1000000 tickets).

Problem 3.60. *1. A bus ticket (see the previous Exercise) is deemed to be extremely lucky if the last three digits form a permutation of its initial digits. Here are some examples of such numbers:* 013103, 225252, 300030, 112112, 112211, 777777. *How many extremely lucky tickets are there?*

2. A bus ticket is considered to be incredibly lucky if the last three digits are the same as three initial digits and they stand in the reverse order. Here are some examples of such numbers: 111111, 002200, 175571, 988889. *How many incredibly lucky tickets are there?*

Answer.

1. 5140;

2. 1000.

Solution. 1. An extremely lucky number can be composed of one, two, or three different digits. The simplest case is when there is only one digit: there are only 10 numbers of interest in this case. Now, count the extremely lucky numbers that can be created with two digits a and b. First, assume that the digit a is present in the initial triplet of digits twice (hence, there are two such digits in the right triplet as well) and the digit b is present in one position only. There are 3 options for both triplets (*aab*, *aba* and *baa*), which can be combined arbitrarily (e.g. *aababa*, *abaaba*, *baaaba* etc.). Overall, there are $3 \cdot 3 = 9$ combinations. We get 9 more numbers if we change the roles of a and b. So, there are 18 numbers in total. And this amount of numbers can be created by any two digits. Two digits can be selected in $C_{10}^2 = 45$ ways. Hence, there are $45 \cdot 18 = 810$ numbers with two different digits. Finally, let us count the amount of numbers composed of three different digits. There are $C_{10}^3 = 120$ ways to choose three digits. There are six possibilities to line up any chosen triplet. The order of digits in the left group can be arbitrarily combined with the order of digits in the right group. We conclude that overall there are $120 \cdot 6 \cdot 6 = 4320$ extremely lucky numbers consisting of three different digits. Summing up the above: there are $10 + 810 + 4320 = 5140$ tickets, which are extremely lucky, out of the total amount of 1000000 tickets. This equals about half percent of the total amount of tickets in one series.

Problem 3.61. *1. How many three-digit numbers have the sum of their digits equal to k $(1 \leq k \leq 9)$?*

2. Establish a bijection between three-digit numbers, the digits of which sum up to k and three-digit numbers, the digits of which sum up to $28-k$.

3. Basing on two previous paragraphs, find the amount of three-digit numbers, the digits of which sum up to s $(19 \leq s \leq 27)$.

4. For every natural k from the interval from 10 to 18 inclusive, count the amount of three-digit numbers, the digits of which sum up to k.

Answer.
1. C^2_{k+1}*;*
3. C^2_{29-s}*;*
4. $C^2_{k+1} - C^2_{k-8} - 2 \cdot C^9_{k-9}$.

Solution. This problem differs from 3.58 in that here we talk about three-digit numbers, the first digit of which is non-zero, and not about arbitrary groups of three digits, which might begin with zero.

Finding answers to some of the questions of the current problem, one can use the results of 3.58. However, we choose completely self-sufficient algorithm.

1. If a, b and c are consecutive digits of a number and $a+b+c=k$, k is less than or equal to 9, then $(a;b;c)$ is an integer non-negative solution to the equation

$$x+y+z=k, \tag{3.22}$$

The first component of which is non-zero. Conversely, if $(p;q;r)$ is an integer non-negative solution to equation (3.22), with $p \neq 0$, then p, q and r are consecutive digits of some three-digit number, which has the sum of its digits equal to k. The latter results from the fact that none of the numbers p, q and r can be greater than 9. Thus, there is a bijection between three-digits numbers, the digits of which sum up to k $(1 \leq k \leq 9)$ and those integer non-negative solutions to equation (3.22) that have non-zero first component. Hence, 3.34 Exercise 34 provides the answer to the first question: C^2_{k+1}.

2. Let $\overline{\alpha\beta\gamma}$ be a three-digit number (α, β, γ is its digits, hence, $\alpha \geq 1$). We match it with the number $\overline{xyz}$, the digits of which is relate to the digits α, β, γ as follows: $x = 10-\alpha$, $y = 9-\beta$, $z = 9-\gamma$. We underline that the numbers x, y and z are one-digit and $x \geq 1$ (as $\alpha \leq 9$), hence, any number $\overline{\alpha\beta\gamma}$ has the three-digit number $\overline{xyz}$ corresponding to it by our rule. Conversely, the latter number has $\overline{\alpha\beta\gamma}$ corresponding to it, because $\alpha = 10-x$, $\beta = 9-y$, $\gamma = 9-z$. Thus, our law of correspondence splits all three-digit numbers into pairs. Two numbers forming a pair correspond to each other. In particular, this evidence that this correspondence is bijective. Now, we calculate the sum of digits of the number $\overline{xyz}$, assuming that the sum of digits of the number $\overline{\alpha\beta\gamma}$ equals to k $\quad(\alpha+\beta+\gamma=k)$. We get: $x+y+z= \quad =(10+9+9)-(\alpha+\beta+\gamma)=28-k$. This equality means that if one number of a pair has the sum of its digits equal to k, then for the other number it equals to $28-k$. Therefore, the amounts of numbers with the above sums of their digits are the same.

3. If s is a number from the interval from 19 to 27, then the number $28-s$ belongs to the interval from 1 to 9. According to the first paragraph, C^2_{29-s} numbers have the sum of their digits equal to $28-s$. And the second paragraph evidence that there is the same amount of numbers, the sum of digits of which is s.

4. First, we note that it suffices to count the amounts of numbers, the sums of digits of which equals $10, 11, 12, 13$ and 14. For other values of sums (15 to 18) the result follows straight from the fact derived in the second paragraph.

The amount of three-digit numbers, the sum of digits of which is $10 + d$ ($d = 0, 1, 2, 3, 4$), is the same as the amount of those solutions to the equation

$$x+y+z = 10+d \tag{3.23}$$

(integer non-negative), that posses two following properties:

1. the first component (the value of x) is non-zero;
2. there are no components greater than 9.

Satisfying the first property, equation (3.23) has C^2_{11+d} solutions (see Problem 3.34). It remains to subtract from this number the amount of those solutions that have a component greater than 9 (besides, there can be only one such component in any solution as the right-hand side does not exceed 14).

Those solutions to equation (3.23) that have their first component greater than 9 are in bijective correspondence with that integer non-negative solutions to the equation

$$x+y+z = d+1 \tag{3.24}$$

that have non-zero first component. We reach a bijection by increasing the first component of solutions to equation (3.24) by 9. According to 3.34, there are C^2_{d+2} such solutions.

Those solutions to equation (3.23) that have their second component greater than 9 are in bijective correspondence with that integer non-negative solutions to the equation

$$x+y+z = d$$

that have non-zero first components. We reach a bijection by increasing the second component of solutions to the above equation by 10.

The same feature is intrinsic to those solutions to equation (3.23) that have their third component greater than 9. So in total, there are $2C^2_{d+1}$ solutions to equation (3.23).

It is time to sum up:

If $0 \le d \le 4$, then there are

$$C^2_{11+d} - C^2_{d+2} - 2C^2_{d+1}$$

three-digit numbers, which have the sum of their digits equal to $10+d$.

Note that the formula is valid for $d = 5, 6, 7, 8$, as well, although it is not of great necessity, thanking a bijection form paragraph 2).

In conclusion, we provide the complete table 3.6 of results concerning the sums of digits of three-digit numbers.

Reasonableness Check. Overall, there are 900 three-digit numbers. They are sorted into 27 groups according to the sums of their digits. If the total of all numbers in all groups is 900, then this is strong evidence of the validity of the above calculations. Indeed, the random coincidence is very unlikely. As we can see, the wanted equality holds:

$$2\cdot(1+3+6+10+15+21+28+36+45+54+61+66+69)+70 = 900.$$

Table 3.6. Sums of digits of three-digit numbers

Sum of digits	1 27	2 26	3 25	4 24	5 23	6 22	7 21	8 20	9 19	10 18	11 17	12 16	13 15	14
Amount of numbers	1	3	6	10	15	21	28	36	45	54	61	66	69	70

Problem 3.62. *There are ks rooks of the same color on an $n \times n$ chessboard: s rooks in each of k rows and k rooks in each of s columns. How many such positioning of the rooks exist?*

Answer. $C_n^k \cdot C_n^s$.

Problem 3.63. *Four different pieces (pawn, rook, knight, and bishop) are placed on an $n \times n$ chessboard: two pieces in two rows and two columns each. How many ways are there to place the pieces like that?*

Answer. $(C_n^2)^2 \cdot 4!$.

Problem 3.64. *Tuples with Repetition. Assume we have a large enough amount of symbols (letters) "a" and "b". Placing them next to each other we get tuples of the form "aaaa", "abbb", "abababa", where the letters can repeat multiple times. Such tuples are called tuples with repetition. The length of a tuple is the number of positions in it. For instance, the tuples "aaaa" and "abba" have length* 4 *and the length of the tuple "bbababbb" is* 8. *Tuples of length n are called n-tuples.*

1. *How many different n-tuples can be created with two symbols?*
2. *How many of them have one of the symbols repeating k times and the other $n-k$ times $(0 \le k \le n)$?*

Answer.

1. 2^n;
2. C_n^k.

Clarification.

1. here are two options for each position: the first symbol or the second.
2. A tuple is completely defined when we specify k positions out of available n, in which the first symbol is placed.

Problem 3.65. *Consider tuples with repetition constructed with s given symbols.*

1. *How many tuples of length n can be created?*

2. *How many n-tuples contain k_1 instances of the first symbol, k_2 instances of the second, and so on up to the s-th symbol that appears k_s times in a tuple $(k_1+k_2+...+k_s=n)$?*

Answer.

1. s^n;

2. $\frac{n!}{k_1!k_2!...k_s!}$.

Solution. First Approach. Let $a_1, a_2, ..., a_s$ be the s symbols forming tuples. There are k_1 copies of the symbol a_1, k_2 copies of the symbol a_2, and so on. Finally, there are k_s copies of the symbol a_s. The following procedure of construction of a tuple can be suggested. Take a string split into n squares (cells, positions). First, choose k_1 of them to put the symbol a_1 in them. There are $C_n^{k_1}$ ways to accomplish this first step. The next step is to choose k_2 positions out of remaining $n-k_1$ for the symbol a_2. There are $C_{n-k_1}^{k_2}$ ways to make this, independently of the exact choice of k_1 positions in the first step. Moving step by step, we choose the positions for the symbols $a_3, a_4, ..., a_s$. By virtue of the combinatorial rule of product, there are

$$C_n^{k_1} \cdot C_{n-k_1}^{k_2} \cdot C_{n-k_1-k_2}^{k_3} \cdot ... \cdot C_{n-k_1-k_2-...-k_{s-1}}^{k_s}$$

different tuples of the wanted type. Applying to each factor of the above product the computational formula

$$C_m^r = \frac{m!}{r!(m-r)!},$$

we get the answer stated above.

Second Approach. Again, there are k_1 copies of the symbol a_1, k_2 copies of the symbol a_2, and so on. Finally, there are k_s copies of the symbol a_s. Let us mark the copies of every symbol with additional labels temporarily, so that they differ from each other. We get n symbols, and each of them differs from the others. Consider different permutations of these symbols. There are $n!$ of them. Choose one such permutation. We have symbols a_1, a_2, ..., a_s standing in exact known positions. Now, all permutations that have the symbols a_1, a_2, ..., a_s in the same positions, can be classified as belonging to one group. Let us call any permutations that belong to the same group homogeneous. All of them have the symbols $a_i\,(i=1,2,...,s)$ in the same positions. How many permutations are there in one group? The type of permutation does not change if replace any symbols a_1 with each other, symbols $a_2$with each other, and so on up to symbols a_s. As there are $k_1!$ ways to “shuffle” symbols a_1, $k_2!$ ways to “shuffle” symbols a_2 and so on up to symbols a_s, any group of homogeneous permutations contains $k_1!k_2!...k_s!$ permutations.

Now we can remove our temporary labels to have all homogeneous permutations turned into the same tuple. Non-homogeneous permutations turn into different tuples. Thus, there are

$$\frac{n!}{k_1!k_2!...k_s!}$$

tuples.

5. Properties of Binomial Coefficients C_n^k

1. For any numbers n and k ($n \geq 1$; $k = 0, 1, \ldots, n$), the following equality holds

$$C_n^{n-k} = C_n^k. \tag{3.25}$$

This equality means that in the sequence

$$C_n^0,\ C_n^1,\ C_n^2,\ \ldots,\ C_n^{n-1},\ C_n^n$$

the numbers equidistant from its ends are the same:

$$C_n^n = C_n^0, \quad C_n^{n-1} = C_n^1, \quad C_n^{n-2} = C_n^2, \ldots$$

Why?
First of all, this comes from the computational formula:

$$C_n^k = \frac{n!}{k!(n-k)!},\ C_n^{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}.$$

However, combinatorial proof featuring modeling is much more attractive and instructive.

Equality (3.25) evidence that an n-element set contains the same amount of $(n-k)$-element and k-element subsets. If we manage to verify this directly, without counting, then we will obtain another proof of equality. It is easy to guess that a bijection can be established between the complements of two types of subsets: we match those two subsets, namely k-element subset and $(n-k)$-element one, the union of which is the whole n-element set. Below, there is an example of such correspondence in the case $n = 5,\ k = 2$.

Let $M = \{a, b, c, d, e\}$ be a five-element set. Below we present the pairs of two- and three-element sets matched by complements: 2-element and 3-element subsets

$$\begin{array}{ccc}
\{a, b\} & \leftrightarrow & \{c, d, e\} \\
\{a, c\} & \leftrightarrow & \{b, d, e\} \\
\{a, d\} & \leftrightarrow & \{b, c, e\} \\
\{a, e\} & \leftrightarrow & \{b, c, d\} \\
\{b, c\} & \leftrightarrow & \{a, d, e\} \\
\{b, d\} & \leftrightarrow & \{a, c, e\} \\
\{b, e\} & \leftrightarrow & \{a, c, d\} \\
\{c, d\} & \leftrightarrow & \{a, b, e\} \\
\{c, e\} & \leftrightarrow & \{a, b, d\} \\
\{d, e\} & \leftrightarrow & \{a, b, c\}.
\end{array}$$

These two columns explicitly and unequivocally validate the equality

$$C_5^2 = C_5^3.$$

Construct similar columns and apply them to prove the equality $C_5^1 = C_5^4$.

2. For any numbers n and k ($n \geq 2$; $k = 1, 2, \ldots, n-1$) the following equality holds:

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k. \tag{3.26}$$

For instance, $C_5^3 = C_4^2 + C_4^3$, $C_2^1 = C_1^0 + C_1^1$. Applying the formula

$$\begin{aligned}
C_{n-1}^{k-1} + C_{n-1}^{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} = \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \left(\frac{1}{n-k} + \frac{1}{k} \right) = \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \cdot \frac{n}{(n-k)\cdot k} = \\
&= \frac{n!}{k!(n-k)!} = C_n^k.
\end{aligned}$$

one can easily prove the above equality. However, it is much more important to understand its combinatorial sense. The equality $C_5^3 = C_4^2 + C_4^3$ means that a five-element set has the same amount of three-element subsets (the left-hand side) as the amount of two-element subsets and three-element subsets of a four-element set (the right-hand side). How can this fact be explained? Consider a five-element set

$$M = \{a, b, c, d, e\}.$$

Let us split all its three-element subsets into two groups. The first group will contain all subsets that include the element "a", and the second will be composed of all other subsets. How many subsets are there in each group? In order to get a subset of the first type, one has to add the element a to a two-element subset of the set $\{b, c, d, e\}$. Thorough consideration of this correspondence reveals that there is a bijection between subsets of the first group and two-element subsets of the set $\{b, c, d, e\}$ (produce the corresponding list of pairs to illustrate it). Thus, there are C_4^2 subsets in the first group. The second group is composed of three-element subsets of the set $\{b, c, d, e\}$ and only of them. Hence, it has C_4^3 subsets. The above arguments prove the equality

$$C_5^3 = C_4^2 + C_4^3$$

without resorting to direct calculation. Replacing 5 by n and 3 by k, we get the combinatorial proof of the general equality (3.26). Reproduce it from scratch in all details.

Formula (3.26) is called recursive, because it expresses the binomial coefficient C_n^k with the lower index n through the binomial coefficients C_{n-1}^{k-1} and C_{n-1}^{k} with smaller lower indices. The formula says that having calculated C_{n-1}^{k-1} and C_{n-1}^{k} one can calculate C_n^k.

Recursive formula (3.26) expresses the most essential property of the table of binomial coefficients, which is usually called Pascal's triangle.

$$\begin{array}{ccccccccccc}
 & & & & & C_0^0 & & & & & \\
 & & & & C_1^0 & & C_1^1 & & & & \\
 & & & C_2^0 & & C_2^1 & & C_2^2 & & & \\
 & & C_3^0 & & C_3^1 & & C_3^2 & & C_3^3 & & \\
 & C_4^0 & & C_4^1 & & C_4^2 & & C_4^3 & & C_4^4 & \\
C_5^0 & & C_5^1 & & C_5^2 & & C_5^3 & & C_5^4 & & C_5^5 \\
\multicolumn{11}{c}{\dotfill}
\end{array} \tag{3.27}$$

The dots at the bottom part of the triangle (3.27) mean that it can be extended to infinity. The numbers C_n^0 and C_n^n $(n = 0, 1, 2, 3, ...)$ standing on the sides of the triangle, equal to 1. Every other number equals the sum of two numbers standing right above it in the previous row. These two properties (the sides of ones and recursive formula (3.26)) completely define all values in the triangle. Using these properties, we arrive at:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \multicolumn{11}{c}{\dots\dots\dots\dots\dots\dots} \end{array} \tag{3.28}$$

We see that Pascal's triangle has vertical symmetry axis. This is the manifestation of property (3.25).

3. The following equality holds for any natural n:

$$C_n^0 + C_n^1 + C_n^2 + ... + C_n^{n-1} + C_n^n = 2^n. \tag{3.29}$$

Let us enumerate the rows of Pascal's triangle with numbers 0, 1, 2, ... Then equality (3.29) says that the sum of all numbers of the n-th row of Pascal's triangle equals 2^n.

Equality (3.29) is transparent from a combinatorial point of view: both its sides express the amounts of different subsets of n-element set. The right-hand side is the result of counting of subsets altogether and in the left-hand side one-element, two-element, etc., sets are counted separately.

4. If $n \geq 1$, then

$$C_n^0 - C_n^1 + C_n^2 - C_n^3 + ... + (-1)^{n-1} C_n^{n-1} + (-1)^n C_n^n = 0. \tag{3.30}$$

Let us write down two exact equalities of this type: for $n = 5$ and $n = 6$.

$$C_5^0 - C_5^1 + C_5^2 - C_5^3 + C_5^4 - C_5^5 = 0, \tag{3.31}$$

$$C_6^0 - C_6^1 + C_6^2 - C_6^3 + C_6^4 - C_6^5 + C_6^6 = 0, \tag{3.32}$$

which are equivalent to

$$1 - 5 + 10 - 10 + 5 - 1 = 0 \tag{3.33}$$

and

$$1 - 6 + 15 - 20 + 15 - 6 + 1 = 0. \tag{3.34}$$

As we can see, the equalities (3.31), (3.33) are correct, and in addition, it does not require any special proof, because it is a corollary of the property (3.25) that has been proved above. As to equalities (3.32), (3.34), these equalities are also correct but the reasoning behind this fact is hidden deeper.

Turn back to the general case (3.30). Which combinatorial fact is expressed by this equality? Here it is: any non-empty set has the same amounts of odd-element and even-element subsets. This interpretation becomes explicit, when we rearrange equality (3.30) as follows:

$$C_n^0 + C_n^2 + C_n^4 + C_n^6 + ... = C_n^1 + C_n^3 + C_n^5 + C_n^7 + ... \tag{3.35}$$

(the equality expands on both sides while the upper index does not exceed the lower).

Why does equality (3.35) always hold?

Let us attempt establishing a bijection between even-element and odd-element subsets of an arbitrary (finite) non-empty set M. We choose its arbitrary element a and introduce the following notation:

Π – is the collection of all those even-element subsets of the set M which does not contain a;

Π_a – is the collection of all those even-element subsets of the set M which contain a;

H – is the collection of all those odd-element subsets of the set M which does not contain a;

H_a – is the collection of all those odd-element subsets of the set M which does not contain a.

Proving equality (3.35) is equivalent to proving the equality

$$|\Pi \cup \Pi_a| = |H \cup H_a|,$$

which in its turn is equivalent to establishing a bijection between $\Pi \cup \Pi_a$ and $H \cup H_a$.

Adding the element a to all subsets from Π, we get all subsets of H_a. Similarly, we get all subsets of Π_a upon the addition of the element a to every subset of the H. Thus, there are bijections $\Pi \leftrightarrow H_a$ and $\Pi_a \leftrightarrow H$, and therefore,

$$\Pi \cup \Pi_a \leftrightarrow H_a \cup H.$$

is also a bijection. Formula (3.35) is proved.

In order to make the above proof completely clear, it is worth illustrating it with an exact example. Let $n = 4$, and denote the elements of the set as follows:

$$M = \{a, b, c, d\}.$$

Detaching element a, we get:

$$\Pi = \{\emptyset, \{b, c\}, \{b, d\}, \{c, d\}\};$$

$$\begin{aligned} &\Pi_a = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c, d\}\}; \\ &H = \{\{b\}, \{c\}, \{d\}, \{b, c, d\}\}; \\ &H_a = \{\{a\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}. \end{aligned}$$

A bijection between Π and H_a:

$$\begin{aligned} &\emptyset \leftrightarrow \{a\} \\ &\{b, c\} \leftrightarrow \{a, b, c\} \\ &\{b, d\} \leftrightarrow \{a, b, d\} \\ &\{c, d\} \leftrightarrow \{a, c, d\} \end{aligned}$$

A bijection between H and Π_a:

$$\begin{aligned} &\{b\} \leftrightarrow \{a, b\} \\ &\{c\} \leftrightarrow \{a, c\} \\ &\{d\} \leftrightarrow \{a, d\} \\ &\{b, c, d\} \leftrightarrow \{a, b, c, d\}. \end{aligned}$$

Problems

Problem 3.66. *How many even-element sets are there in an n-element set* $(n \geq 1)$*?*

Problem 3.67. *Applying the major recurrence relation, construct Pascal's triangle (find the values of all its constituents) up to the tenth row inclusive.*

Problem 3.68. *Calculate the sum*

$$C_1^1 + C_2^1 + C_3^1 + C_4^1 + \ldots + C_n^1.$$

Problem 3.69. *Answer to the previous problem is* $\frac{n(n+1)}{2}$*, which is* C_{n+1}^2*. Therefore,*

$$C_1^1 + C_2^1 + C_3^1 + C_4^1 + \ldots + C_n^1 = C_{n+1}^2.$$

Prove this equality: 1) using basic recursive formula (begin with the right-hand side); 2) building combinatorial model (the right-hand side is the amount of two-element subsets of an $(n+1)$*-element set; how to classify these subsets so that the result is the left-hand side of the equality?)*

Problem 3.70. *Calculate (reduce) the sum*

$$C_2^2 + C_3^2 + C_4^2 + C_5^2 + \ldots + C_n^2.$$

Hint. Use the Pascal's triangle.

Problem 3.71. *Prove the major recurrence relation*

$$C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$$

using mathematical induction technique with respect to k.

Problem 3.72. *Construct combinatorial model (or models) to prove the equality*

$$C_{n+m}^k = C_n^0 \cdot C_m^k + C_n^1 \cdot C_m^{k-1} + C_n^2 \cdot C_m^{k-2} + \ldots + C_n^{k-1} \cdot C_m^1 + C_n^k \cdot C_m^0$$

(the Cauchy equality).

Hint. Consider arbitrary $(n+m)$-element subset of the set M and devode M into two parts A and B of n and m elements respectively. The left-hand side of the equality is the amount of k-element subsets of the set M. The summand $C_n^s \cdot C_n^{k-s}$ in the right-hand side expresses the amount of k-element subsets of the set M that contain s elements from the part A and $k-s$ elements from the part B.

Problem 3.73. *From the Cauchy equality (see the previous problem), deduce the following equality:*

$$C_{2n}^n = (C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 + \ldots + (C_n^n)^2.$$

Problem 3.74. *Prove the equality*

$$C_n^m \cdot C_m^k = C_n^k \cdot C_{n-k}^{m-k}$$

$(n \geq m \geq k)$ *in at least two ways: 1) formally-arithmetically; 2) constructing an appropriate combinatorial model.*

Hint. One possible model. Let C be an n-element set. Consider all possible pairs of sets $(A;B)$, where: $|A|=k$, $|B|=m$ and $A\subset B\subset C$. Now, it suffices to come up with two different methods of counting of such pairs, where one yields the amount $C_n^m\cdot C_m^k$, and another one yields the amount $C_n^k\cdot C_{n-k}^{m-k}$.

Problem 3.75. *Prove the following equality in two different ways*

$$C_k^k+C_{k+1}^k+C_{k+2}^k+\ldots+C_n^k=C_{n+1}^{k+1}.$$

Hint.

1) Compare the left-hand side of the equality with the sums from problems 3.68 and 3.70. Refer to Pascal's triangle.

2) The equality can be proved by induction on n for a fixed k (the basis of induction is $n=k$).

3) Possible combinatorial model. Consider the set

$$X=\{1,2,3,\ldots,k,k+1,\ldots,n,n+1\}.$$

The equality is about its $(k+1)$-element subsets. Classify them as follows:

1. The subsets that contain 1.
2. The subsets that do not contain 1, but contain 2.
3. The subsets that do not contain 1 and 2, but contain 3.

And so on.

Finally, $n-k+1$. The subsets that do not contain $1,2,\ldots,n-k$, but contain $n-k+1$.

Problem 3.76. *Split all n-element subsets of the sets*

$$A=\{1,2,3,\ldots,n,n+1,\ldots,n+r+1\}$$

into the following groups:

1. The subsets that do not contain 1.

2. The subsets that contain 1, but do not contain 2.

3. The subsets that contain 1 and 2, but do not contain 3.

4. The subsets that contain 1, 2 and 3, but do not contain 4.

And so on.

n. The subsets that contain 1, 2, 3, …, $n-1$, but do not contain n.

n+1. The subsets that contain 1, 2, 3, …, $n-1$, n.

How many subsets are there in every group?

Prove that every n-element subset of the set A belongs to exactly one of the groups.

How many n-element subsets does the set A have?

The answers to the previous questions enable one to write down certain equation for binomial coefficients. What equation is it?

Problem 3.77. *Prove the equality*

$$C_p^s=C_{p-q}^0\cdot C_q^s+C_{p-q}^1\cdot C_1^{s-1}+C_{p-q}^2\cdot C_q^{s-2}+\ldots+C_{p-q}^s\cdot C_q^0.$$

Hint. Compare this equality with the Cauchy equality from Problem 3.72.

Problem 3.78. *Prove the equality*

$$nC_n^r = (r+1)C_n^{r+1} + rC_n^r$$

a) arithmetically; b) building an appropriate combinatorial model.

Hint. The following model can be suggested. Let A be a set of n elements. Consider all possible pairs $(t;x)$, where t is an element of A and X is an r-element subset of the set A. How many such pairs exist? There are $n \cdot C_n^r$ (n ways to choose t and C_n^r ways to choose a subset of A). Which two groups (by which feature) should these pairs be split into, in order to obtain the wanted equality?

Problem 3.79. *Consider n-element set A and all possible pairs $(X;Y)$, where $X \subset A$, $Y \subset A$, and $|X| = 2$, $|Y| = k$.*

How many such pairs exist?

Split all these pairs into three groups depending on the amount of common elements of the subsets X and Y: zero, one, or two.

Ensure that the first group $(X \cap Y = \emptyset)$ contains $C_n^{k+2} \cdot C_{k+2}^2$ pairs, the second group contains $2 \cdot C_n^{k+1} \cdot C_{k+1}^2$ pairs, and the third consists of $C_n^k \cdot C_k^2$ pairs. Which equality is the result of the above calculations?

Problem 3.80. *Apply the experience gained in two previous problems to build the combinatorial model of the equality*

$$\begin{gathered} C_n^s \cdot C_n^r = C_s^0 \cdot C_{r+s}^r \cdot C_n^{r+s} + C_s^1 \cdot C_{r+s-1}^{r-1} \cdot C_n^{r+s-1} + \\ + C_s^2 \cdot C_{r+s-2}^{r-2} \cdot C_n^{r+s-2} + \ldots + C_s^q \cdot C_{r+s-q}^{r-q} \cdot C_n^{r+s-q} \end{gathered}$$

$$(q = \min\{r,s\}).$$

Problem 3.81. *Prove the equality*

$$1 \cdot C_n^1 + 2C_n^2 + 3 \cdot C_n^3 + \ldots + n \cdot C_n^n = n \cdot 2^{n-1},$$

applying the formula

$$C_m^k = \frac{m}{k} C_{m-1}^{k-1}$$

to every summand in its left-hand side. Build a combinatorial model for it.

Hint. An example of the combinatorial model. Let A be an n-element set. Consider all possible pairs $(t;X)$, where t is an element of A and X is a subset of A, $t \notin X$ (t does not belong to the subset X). The number on the right-hand side is the amount of all such pairs. Proceed with the splitting of these pairs into groups, depending on the amount of elements in X, and count the amount of pairs in each group.

Problem 3.82. *Prove the equality*

$$1^2 \cdot C_n^1 + 2 \cdot C_n^2 + 3^2 \cdot C_n^3 + \ldots + n^2 \cdot C_n^n = n \cdot (n+1) \cdot 2^{n-2}.$$

Hint. One can apply the formula

$$C_m^k = \frac{m}{k} C_{m-1}^{k-1},$$

to every summand in the left-hand side of the wanted equality, and then use the equality from the previous problem). Also, attempt to construct the combinatorial model of equality. Suggested idea: consider pairs $([a,b];X)$, where $[a,b]$ are 2-tuples with repetition constructed of the elements of an n-set A, and X is a subset of this set that does not include a and b.

Problem 3.83. *Calculate the sum*

$$C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \ldots + \frac{1}{n+1}C_n^n.$$

Hint. Denoting the sum by s, begin with calculation of $(n+1)\cdot s$.

Problem 3.84. *Calculate the sum*

$$C_n^1 - 2C_n^2 + 3C_n^3 - \ldots + (-1)^{n-1} n C_n^n.$$

Hint. Apply the formula $C_m^k = \frac{m}{k} C_{m-1}^{k-1}$ *to every summand.*

Problem 3.85. *Prove the equalities*
a) $C_n^1 + 3C_n^3 + 5C_n^5 + \ldots = n \cdot 2^{n-2}$
and
b) $C_n^2 + 2C_n^4 + 3C_n^6 + \ldots = n \cdot 2^{n-3}$,
applying the results of problems 3.81 and 3.84 (in both cases, the left-hand sides contain all the summands, where the upper index does not exceed the lower). Construct combinatorial models for both equalities.

Problem 3.86. *Applying the recursive formula* $C_n^0 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^m C_n^m$, *calculate (reduce) the sum*

$$C_n^0 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^m C_n^m.$$

Problem 3.87. *Calculate the sum*

$$\frac{1}{2}C_n^1 - \frac{1}{3}C_n^2 + \frac{1}{4}C_n^3 - \ldots + (-1)^{n+1} \cdot \frac{1}{n+1} C_n^n.$$

Answer. $\frac{n}{n+1}$.

Hint. You may apply the formula

$$C_n^k = \frac{k+1}{n+1} C_{n+1}^{k+1}$$

to each summand.

Problem 3.88. *Prove the equality*

$$C_n^{n-m}\cdot C_n^0 - C_{n-1}^{n-m}\cdot C_n^1 + C_{n-2}^{n-m}\cdot C_n^2 - C_{n-3}^{n-m}\cdot C_n^3 + \ldots + (-1)^m\cdot C_{n-m}^{n-m}\cdot C_n^m = 0$$
$$(0 < m \le n).$$

Reduced form:

$$\sum_{k=0}^{m}(-1)^k\cdot C_{n-k}^{n-m}\cdot C_n^k = 0.$$

Hint. Take into account the equality

$$C_{n-k}^{n-m} = C_{n-k}^{m-k},$$

and apply it to every summand of the equality from problem 3.74.

Problem 3.89. *Calculate (reduce) the sum*

$$\sum_{k=0}^{m} C_{n-k}^{n-m}\cdot C_n^k.$$

Answer. $2^m\cdot C_n^m$.

Hint. Apply the experience of the previous problem.

Problem 3.90. *Prove that*

$$\sum_{k=0}^{n-m}(-1)^k\cdot C_{n-k}^{m}\cdot C_n^k = 0 \quad (0 \le m < n).$$

Extended form:

$$C_n^m\cdot C_n^0 - C_{n-1}^m\cdot C_n^1 + C_{n-2}^m\cdot C_n^2 - \ldots + (-1)^{n-m}\cdot C_n^{n-m} = 0.$$

Hint. Compare the above equality with the equality from problem 3.88.

Problem 3.91. *Calculate (reduce) the sum*

$$\sum_{k=0}^{n-m} C_{n-k}^{m}\cdot C_n^k \quad (0 \le m \le n).$$

Hint. This problem can be reduced to problem 3.89.

Problem 3.92. *Prove that*

$$\sum_{k=0}^{m} C_n^k C_n^{m-k} = C_{2n}^m \quad (m \le n).$$

Hint. Compare the above equality with the Cauchy equality (problem 3.72). In addition, build a combinatorial model.

Problem 3.93. *Prove (e.g., by induction) the equality*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}C_n^k = \sum_{k=1}^{n}\frac{1}{k}.$$

Extended form:

$$C_n^1 - \frac{1}{2}C_n^2 + \frac{1}{3}C_n^3 - \ldots + (-1)^{n-1}\cdot\frac{1}{n}\cdot C_n^n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}).$$

6. Raising Binomials to Powers. Newton's Binomial Formula

It may seem that we have always known the formula of raising the sum of two terms to the power of 2:

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{3.36}$$

If necessary, we can recall the above formula easily and without any effort. It would seem quite surprised if one used the chain of transformations

$$(a+b)^2 = (a+b)\cdot(a+b) = a\cdot a + a\cdot b + b\cdot a + b\cdot b = a^2 + 2ab + b^2.$$

every time, when it is necessary to get the right-hand side of equality from its left-hand side. We can reproduce the above steps but we are used to skipping the steps involved. The summands in the right-hand side, namely, a^2, $2ab$ and b^2, have reliably imprinted in our memory.

The above concerns to the formula of raising the sum of two terms to the power 3 as well:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \tag{3.37}$$

When required, we can easily deduce a similar formula for the fourth power of the above sum. Here is this formula:

$$\begin{aligned} &(a+b)^4 = (a+b)^3\cdot(a+b) = \\ &= (a^3 + 3a^2b + 3ab^2 + b^3)\cdot(a+b) = \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \end{aligned} \tag{3.38}$$

Basing on (3.38), it is straightforward to derive the formula for the fifth power of binomial and so on. We strongly suggest the reader take a while to develop similar formulas for the fifth and sixth powers.

It does not make much sense to spawn this type of formula for the higher powers (seventh, eighth, etc.). It is much more feasible (and interesting) to find out if the rules (3.36), (3.37), (3.38), etc., which are applicable in the special cases of raising of a binomial to the second, third, fourth, etc. power can be generalized for the case of power n.

Guessing the General Rule. In order to detect the common patterns of the formulas of raising of a binomial to powers $2, 3, 4, 5, 6$, we need to write down the list of these formulas in compact form. It is also useful to accompany them with the obvious formula for the case of power 1. The resulting list of formulas is presented below.

$$\begin{aligned} &(a+b)^1 = a + b; \\ &(a+b)^2 = a^2 + 2ab + b^2; \\ &(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3; \\ &(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4; \\ &(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5; \\ &(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6. \end{aligned}$$

Observe that:

1. raising a binomial to the power n, we get $n+1$ summands in the right-hand side;

2. every of the above summand has its combined power (w.r.t. the variables a and b) equal to n;
3. the powers of variable (letter) a decrease by 1 in each consecutive summand, gaining all values from n to 0;
4. the powers of variable (letter) b increase by 1 in each consecutive summand, gaining all values from 0 to n;
5. the coefficients in summands that are equidistant from the ends of the sums are the same.

Regarding the rule that governs the values of coefficients, it is necessary to extract them from the respective summands and express them in a form of a triangle table:

Power 1	1	1					
Power 2	1	2	1				
Power 3	1	3	3	1			
Power 4	1	4	6	4	1		
Power 5	1	5	10	10	5	1	
Power 6	1	6	15	20	15	6	1

We recognize this table to be a part of Pascal's triangle formed by C_n^k. Thus, for $n = 6$, we get the equality:

$$(a+b)^6 = C_6^0 a^6 + C_6^1 a^5 b + C_6^2 a^4 b^2 + C_6^3 a^3 b^3 + C_6^4 a^2 b^4 + C_6^5 ab^5 + C_6^6 b^6.$$

Is there enough evidence to confidently claim that the patterns observed for $n = 1, 2, 3, 4, 5, 6$ are intrinsic to any higher power n? Even more important: raising a binomial to the power n, will the coefficients always coincide with the values from the corresponding row of Pascal's triangle? And if the above is true, what is the reasoning behind this?

Proof of the General Rule. To raise the binomial $a+b$ to power n means to multiply $a+b$ by $a+b$, then multiply the result by $a+b$, and so on, until the factor $a+b$ repeats n times. Imagine that we have performed this time-consuming procedure with no simplifications: no transposition of factors, no expression of the product of the same factors as power, no reduction of the same factors. As a result, we would have obtained the following sequence of equalities:

$$\begin{aligned}
&(a+b)^1 = a+b\\
&(a+b)^2 = (a+b)(a+b) = aa+ba+ab+bb;\\
&(a+b)^3 = (a+b)^2(a+b) = (aa+ba+ab+bb)(a+b) =\\
&aaa+baa+aba+bba+aab+bab+abb+bbb;\\
&(a+b)^4 = (a+b)^3(a+b) =\\
&= (aaa+baa+aba+bba+aab+bab+abb+bbb)(a+b) =\\
&= aaaa+baaa+abaa+bbaa+aaba+baba+abba+bbba+\\
&+aaab+baab+abab+bbab+aabb+babb+abbb+bbbb;
\end{aligned}$$

etc.

Having determined the sum S_k for $(a+b)^k$, we get the sum S_{k+1} for $(a+b)^{k+1}$ by the following three-step rule:

1. attach the letter a to the right ends of all summands of the sum S_k;
2. attach the letter b to the right ends of all summands of the sum S_k;
3. add (connect with the "+" sign) two sums, obtained according to paragraphs 1) and 2).

Hence, raising the binomial $a+b$ to the power n following the above rule, we get the sum of all possible n-tuples composed of letters a and b on the right-hand side. As we already know, there are 2^n such tuples. This is the number of summands in our sum.

Now, we sort the tuples depending on the number of times they contain the letter b (which is the same as for a). We get $n+1$ different types of tuples:
type 0 (tuples that does not contain b);
type 1 (tuples that contain one instance of b);
type 2 (tuples that contain two instances of b);
. . . .
type k (tuples that contain k instances of b);
. . . .
type n (tuples that contain n instances of b).

Once we transform the products into their compact form ($t^s = t \cdot t \cdot t \cdot \ldots \cdot t$ (s times)), the following occurs:
all type 0 tuples turn into a^n;
all type 1 tuples turn into $a^{n-1}b$;
all type 2 tuples turn into $a^{n-2}b^2$;
. . . .
all type k tuples turn into $a^{n-k}b^k$;
. . . .
all type n tuples turn into b^n.

Summing up similar terms, near each summand of the form $a^{n-k}b^k$, we get the coefficient, which is equal to the number of tuples that have turned into this term. As it has been discussed above, there are C_n^k $n-k$tuples of type k. The result is the following formula:

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^n b^n, \qquad (3.39)$$

or briefly:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k.$$

The symbol $\sum\limits_{s=1}^{n} c_s$ denotes the sum $c_1 + c_2 + c_3 + \ldots + c_n$. For example,

$$\sum_{s=1}^{5} s^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2;$$

$$\sum_{i=0}^{7}\frac{2i+1}{i^2+1}=\frac{1}{1}+\frac{3}{2}+\frac{5}{5}+\frac{7}{10}+\frac{9}{17}+\frac{11}{26}+\frac{13}{37}+\frac{15}{50};$$

$$\sum_{k=0}^{4}(-1)^k(k+1)=1-2+3-4+5.$$

Symmetrical (w.r.t. a and b) equality (3.39) is called Newton's binomial formula (although it was known before him, and its special cases (3.36) and (3.37) were studied in schools of ancient Babylon at least 4000 years ago; however, Newton developed its generalization).

There are two cases of formula (3.39) that the reader ought to develop excellent command of (along with the command of "major case" (3.39), of course).

1. Replacing in (3.39) "b" for "$-b$", one gets:

$$\begin{aligned}(a-b)^n=\sum_{k=0}^{n}(-1)^kC_n^ka^{n-k}b^k=\\ =C_n^0a^n-C_n^1a^{n-1}b+C_n^2a^{n-2}b^2-C_n^3a^{n-3}b^3+\ldots\\ +(-1)^kC_n^ka^{n-k}b^k+\ldots+(-1)^nC_n^nb^n.\end{aligned} \tag{3.40}$$

1. For $a=1$, $b=x$, formula (3.39) transforms into

$$(1+x)^n=\sum_{k=0}^{n}C_n^kx^k=C_n^0+C_n^1x+C_n^2x^2+\ldots+C_n^kx^k+\ldots+C_n^nx^n. \tag{3.41}$$

The results presented above completely clarify the nature of name "binomial coefficients", which is used for C_n^k.

Problems

Problem 3.94. *Construct Pascal's triangle and use it to raise the following binomials to the given powers:*

1. $(a+b)^6$;
2. $(p-q)^5$;
3. $(1-z)^6$;
4. $(x-1)^4$;
5. $(t^2+1)^4$;
6. $(2-x)^6$.

Problem 3.95. *Write down the general formula for raising the binomial* $1-x$ *to the n-th power.*

Problem 3.96. *Are there any summands without the letter a in the expansion of the expression* $(a^{\frac{1}{17}}+a^{-\frac{1}{7}})^{48}$ *by Newton's binomial formula? If there are, list them without calculation of binomial coefficients. Answer the above questions without computation of 49 summands of the binomial formula.*

Answer. C_{48}^{24}.

Problem 3.97. *Calculate*

$$(\sqrt{x^2+1}+x)^5-(\sqrt{x^2+1}-x)^5.$$

Answer. $32x^5+40x^3+10x$.

Problem 3.98. *The binomial formula can be applied for approximate calculations. We explain this with the following example.*

$$\left(1+\frac{3}{1000}\right)^7=1+7\cdot\frac{3}{1000}+21\cdot\left(\frac{3}{1000}\right)^2+35\cdot\left(\frac{3}{1000}\right)^3+\dots$$

Clearly, the last six of the eight summands on the right-hand side (four of them are replaced by dots) are incomparably smaller than the first two. Removing them (six small summands), one gets the number that is close enough to the true value of the expression:

$$(1,003)^7\approx 1+7\cdot 0,003=1,021.$$

The sign "$\approx$*" reads like "approximately equal". In our case, the true value* $(1,003)^7$ *differs from* $1,021$ *much less than by 0,001. To verify this, it is enough to find an estimate for the third summand* $21\cdot\left(\frac{3}{1000}\right)^2$*, as the other five are much less.*

1. *Find approximate values of the following expressions, considering two initial summands in the respective sums:*

   a) $(1,002)^5$*;*

   b) $(2,001)^{10}$*;*

   c) $(0,998)^8$*;*

   d) $(1,999)^9$.

2. *Evaluate exact and approximate (restricted to two initial terms in respective binomial formulas) values of the following expressions and compare the results:*

a) $(0,999)^3$*;*
b) $(2,02)^3$.

Problem 3.99. *In formulas* (3.39) *and* (3.40)*, one can put* $a=b=1$ *to get interesting equalities involving the binomial coefficients:*

$$C_n^0+C_n^1+C_n^2+C_n^3+\dots+C_n^{n-1}+C_n^n=2^n, \tag{3.42}$$

$$C_n^0 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^{n-1}C_n^{n-1} + (-1)^n C_n^n = 0. \quad (3.43)$$

Combinatorial meaning of the former is that an n-element set contains 2^n different subsets (including an empty set). What is the combinatorial meaning of the latter equality? Use (3.42) and (3.43) to compute

$$C_n^0 + C_n^2 + C_n^4 + C_n^6 + \ldots$$

and

$$C_n^1 + C_n^3 + C_n^5 + C_n^7 + \ldots$$

(the first sum ends with the summand C_n^{2k}, where $2k$ is the greatest even number less than n, and the second ends with C_n^{2k+1}, where $2k+1$ – is the greatest odd number less than n). What combinatorial meaning do these sums have?

Problem 3.100. *Find the values of the following sums:*

1. $C_m^1 + C_m^2 + C_m^3 + \ldots + C_m^{m-1}$;
2. $C_m^1 - C_m^2 + C_m^3 - C_m^4 + \ldots + (-1)^{m-1}C_m^m$;
3. $C_s^2 - C_s^3 + C_s^4 - \ldots + (-1)^s C_s^s$;
4. $C_{2k+1}^0 + C_{2k+1}^1 + C_{2k+1}^2 + \ldots + C_{2k+1}^k$;
5. $C_{4m}^0 - C_{4m}^1 + C_{4m}^2 - C_{4m}^3 + \ldots + C_{4m}^{2m-2} - C_{4m}^{2m-1}$;
6. $C_{4m-2}^0 - C_{4m-2}^1 + C_{4m-2}^2 - C_{4m-2}^3 + \ldots - C_{4m-2}^{2m-3} + C_{4m-2}^{2m-2}$;
7. $C_{2n}^0 + C_{2n}^1 + C_{2n}^2 + \ldots + C_{2n}^{n-1}$.

<u>Answer.</u> *1) $2^m - 2$; 2) 1; 3) $s-1$; 4) 4^k; 5) $-\frac{1}{2}C_{4m}^{2m}$; 6) $\frac{1}{2}C_{4m-2}^{2m-1}$; 7) $2^{2n-1} - \frac{1}{2}C_{2n}^n$.*

<u>Hint.</u> *It is desirable to solve this problem verbally. Compare each sum with equality (3.42) or (3.43).*

Problem 3.101. *Equality (3.43) reads that the amount of even-element subsets of a finite non-empty set is the same as the amount of odd-element ones. Prove this fact by establishing a bijection between the subsets of both types of an n-element set A. You may exploit the familiar idea. Let a be an element of A. Adding the element a to all subsets that do not contain it, one gets all subsets that include it. Take a closer look at this correspondence. Is it the sought correspondence between the even-element and odd-element subsets of the set A?*

Problem 3.102. *Newton's binomial formula is an inexhaustible source of various equalities involving the binomial coefficients C_n^k. Several examples are presented below.*

1. Begin with the equality

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n}.$$

Expand both expressions by the binomial formula, and compare the coefficients at x^n in both sides of the resulting equality. Upon the completion of this task, you will get one of the

most prominent combinatorial equalities, which we have encountered above. This would be the new proof of this equality.

2. What equality should one start with, and what actions should be taken in order to prove the equality

$$C_n^0 \cdot C_m^k + C_n^1 C_m^{k-1} + C_n^2 C_m^{k-2} + \ldots + C_n^k C_m^0 = C_{n+m}^k?$$

Here n, m and k are positive integers, and k does not exceed n and m.

3. Imagine that the only thing you know about C_n^k is that they are the coefficients in formula (4). Explain with reasons if there are any grounds to state that

$$C_n^{n-k} = C_n^k.$$

4. How can the recurrence relation

$$C_{n+1}^k = C_n^k + C_n^{k-1} \tag{3.44}$$

be proved by means of the binomial formula?

5. Use the binomial formula to prove the equality

$$C_{n+2}^k = C_n^{k-2} + 2C_n^{k-1} + C_n^k.$$

Deduce this equality from formula (3.44).

6. Compute the sum

$$C_n^0 + 2C_n^1 + 4C_n^2 + 8C_n^3 + \ldots + 2^n C_n^n.$$

7. Compute the sum

$$C_n^0 - 2C_n^1 + 4C_n^2 - 8C_n^3 + \ldots + (-1)^n \cdot 2^n C_n^n.$$

Answer. 1). $(C_n^0)^2 + (C_n^1)^2 + \ldots + (C_n^n)^2 = C_{2n}^n$.

2). Compute and compare the coefficients at x^k in both sides of the equality

$$(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}.$$

3). The equality in question is the corollary of the commutativity law of addition:

$$(a+b)^n = (b+a)^n.$$

4). In the equality $(1+x)^{n+1} = (1+x)^n \cdot (1+x)$, the coefficients at x^k should be computed.

5). In the equality $(1+x)^{n+2} = (1+x)^n \cdot (1+x)^2$ the coefficients at x^k should be computed.

6). 3^n. 7). $(-1)^n$.

Problem 3.103. *Reduce the sum*

$$z + C_n^1 z^2 + C_n^2 z^3 + \ldots + C_n^n z^{n+1}.$$

Answer. $z \cdot (1+z)^n$.

Problem 3.104. *Prove the equality*

$$C_n^1 + 2C_n^2 + 3C_n^3 + 4C_n^4 + ... + nC_n^n = 2^{n-1} \cdot n.$$

<u>Answer.</u> 2^{n-1}.

<u>Hint.</u> Attempt dividing the equality by n term-wise and closely examine each summand in the left-hand side, which are

$$\frac{k}{n}C_n^k \ (k = 1, 2, ..., n).$$

Problem 3.105. *Prove the equality*

$$C_n^0 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + ... + \frac{1}{n+1}C_n^n = \frac{2^n - 1}{n+1}.$$

<u>Hint.</u> Apply the technique from the previous problem 3.104.

Problem 3.106. *Are there three consecutive binomial coefficients C_n^{k-1}, C_n^k and C_n^{k+1} that are consecutive terms of an arithmetic progression? How many such triplets exist, if any? List the corresponding values of n and k.*

<u>Answer.</u> There are infinitely many triplets, though they exist for special values of n only. n should be less by 2 than the square of some natural number: $n = s^2 - 2$ $(s = 3, 4, 5, 6, ...)$. For every value of $n = s^2 - 2$, there are two three-element arithmetic progressions of the sought type: one is increasing and the other is decreasing and inverse to the first (it consists of the same numbers following in the reverse order). The corresponding values of k:

$$k = \frac{n \pm \sqrt{n+2}}{2},$$

Therefore,
$k = \frac{(s-1)(s+2)}{2}$ *and* $k = \frac{(s+1)(s-2)}{2}$.

Problem 3.107. *No three consecutive binomial coefficients C_n^{k-1}, C_n^k and C_n^{k+1} exist that are the consecutive terms of a geometric progression. Explain this fact.*

<u>Hint.</u> For a fixed n, determine the behavior of $\frac{C_n^{k+1}}{C_n^k}$, when k grows from 0 to $n-1$.

Problem 3.108. *In the expansion of the expression $(3-2x)^n$ by the powers of x (e.g., following the binomial formula), the sum of coefficients of the resulting polynomial is 1. Prove this fact. What is the sum of coefficients in the polynomial $(2x-3)^n$?*

Problem 3.109. *If any two consecutive summands in the binomial decomposition of $(1+c)^n$ are equal (c is non-zero), then $n! \cdot c$ is a positive integer. Why?*

Problem 3.110. *Prove that for any integer k from the interval $[1, n]$, the following inequality holds:*

$$C_{2n+k}^n \cdot C_{2n-k}^n < (C_{2n}^n)^2.$$

<u>Hint.</u> Compare the products $C_{2n+k}^n \cdot C_{2n-k}^n$ and $C_{2n+k+1}^n \cdot C_{2n-k-1}^n$, where $k \in [0, n-1]$.

Problem 3.111. *Are there three consecutive binomial coefficients*

$$C_n^{k-1},\, C_n^k,\, C_n^{k+1},$$

such that

$$C_n^{k-1}+C_n^k=C_n^{k+1}?$$

How many such triplets exist if any? Find all of them, defining the corresponding values of n and k or constructing an algorithm that allows to determine all eligible pairs $(n;k)$.

Information. *This is a problem of serious research rather than an ordinary exercise.*

Answer. *There are infinitely many suitable triplets, but they are very rare;*

$$k=u_{2i-1}\cdot u_{2i}-1 \quad (i=2,3,\ldots),$$

where u_m are the elements of the Fibonacci sequence:

$u_1=1,\ u_2=1;\ u_s=u_{s-1}+u_{s+2}$ *for* $s=3,4,5,\ldots;$

The corresponding values of n:

$$n=\frac{3k+\sqrt{k^2+4(k+1)^2}}{2}.$$

Below we list several triplets with the lowest values of k (and n):

$$C_{14}^4,\ C_{14}^5,\ C_{14}^6 \quad (k=5;\ n=14);$$

$$C_{103}^{38}, C_{103}^{39}, C_{103}^{40} \quad (k=39;\ n=103);$$

$$C_{713}^{271},\ C_{713}^{272},\ C_{713}^{273} \quad (k=272;\ n=713);$$

$$C_{4181}^{1868}, C_{4181}^{1869}, C_{4181}^{1870}\ (k=1869; n=4181).$$

Chapter 4

Paths in a Rectangle

A network of roads constructed in two mutually orthogonal directions is an inexhaustible source of interesting and instructive combinatorial problems, which are the topic of the current chapter. Probably, the most prominent among them is presented below in the first paragraph.

1. Consider an $m \times n$ rectangle split into m vertical and n horizontal stripes by straight lines parallel to its sides. In Fig. 4.1, there is an illustration of the above for $m = 7$ and $n = 4$.

Intersecting pairwise, these stripes create mn (in Fig. 4.1 there are 35 of them) squares with sides of length 1. Imagine that all lines ($m+1$ vertical and $n+1$ horizontal) are paths, following which one can get, say, from the “southwestern” point A to the “northeastern” point B. Departing from the point A with the intention to get to the point B following the shortest possible route, at each intersection, one has to choose where to move next. Is there a rule, following which one can get from A to B using the shortest eligible path and not get lost in the way? Obviously, one such path can be pointed out (e.g., ACB). However, the problem is not about this. What we actually need is to define features of the shortest paths that distinguish them from all other possible routes along the drawn lines. First, we need to find the length of the shortest path from A to B. This is rather simple task. We are allowed to move along the horizontal and vertical lines only, that is, we are limited to two orthogonal directions: “south-north” and “west-east”. The point B is located in m units of length to the east and n units of length to the north from the point A. Thus, in order to get to the destination, we have to cover the distance of m, moving from the west to the east, and the distance of n, moving from the south to the north. This means that the shortest paths from A to B are of length $m+n$. For example, ACB and ADB are two of such paths. But there are other paths. A path from A to B is the shortest if it consists of m horizontal and n vertical elementary (of length 1) intervals (this is how we will call the part of path between two adjacent points of intersection). It does not matter in which order one passes along these intervals. That is why there are many shortest paths from A to B. In Fig. 4.1 two of them are outlined. The first and the most important question about paths in a rectangle concerns to the number of the shortest paths from the point A to the point B.

Problem 4.1. *How many different shortest paths from the point A to the point B, laying along with the described above lines, are there?*

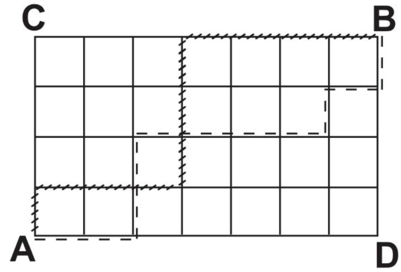

Figure 4.1. Paths in rectangle.

There is another formulation of this problem that can be given with the help of the notion of the coordinate plane.

Let one is allowed to move on the coordinate plane along the integer-valued coordinate lines only (that is, along the lines $x = k$ and $y = s$, where k and s are integer). How many shortest ways from the point $A\,(0;0)$ to the point $B\,(m;n)$ are there?

The paths that are the subject of problem 4.1 and the previous question are polygonal chains, having the following defining properties:

a) their ends are the points A and B;

b) their vertices are integer-valued points (the points with integer coordinates);

c) their line segments are parallel to coordinate axes (sides of a rectangle);

d) each such polygonal chain is the shortest of those that possess the above three properties (is of length $m+n$).

The problem 4.1 can be formulated in such a way that it concerns to the above defined polygonal chains instead of paths from A to B.

Problem 4.2. *Let $A\,(0;0)$ and $B\,(m;n)$ be the points of the coordinate plane (m and n are given natural numbers). How many different polygonal chains of length $m+n$ with the ends A and B have the following properties:*

1. *their vertices have integer coordinates;*
2. *every of their line segments is parallel to one of the coordinate axes?*

The problem 4.1 can be transformed into a clearly arithmetical problem. Moving along the polygonal chain from the point $A\,(0;0)$ to the point $B\,(m;n)$, we get into a point with integer coordinates after every elementary (unit) segment of the path. The sum of coordinates of every next point is greater by 1 than one of the previous point. Therefore, arithmetical interpretation of the path from A to B along the shortest polygonal chain is given by the sequence of pairs of non-negative integer numbers, which begins with the pair $(0;0)$ and ends with the pair $(m;n)$, and any adjacent elements of which have one of their coordinates equal and the other differing by 1 (the pair with a higher sequence number has a greater value of this component). For example, the polygonal chain beginning with the horizontal segment which is highlighted in Fig. 4.1, corresponds to the following sequence of pairs:

$$(0;0) \to (1;0) \to (2;0) \to (2;1) \to (2;2) \to (2;3) \to$$
$$\to (3;3) \to (4;3) \to (5;3) \to (6;3) \to (6;4) \to (7;4) \to (7;5).$$

As we can see, the problem concerning the shortest paths in an $m \times n$ rectangle transforms in the problem about sequences (chains) of pairs of non-negative integer numbers.

Problem 4.3. *Consider chains of pairs of integer numbers, which obey the following rules. The initial pair of a chain is* $(0;0)$*, the next is* $(1;0)$ *or* $(0;1)$*. Each next pair has one of its components greater by 1 than one of the previous pair. The last pair of the chain is* $(m;n)$*.*

How many such chains exist?

Thus, we have three versions of the same problem. It is straightforward to create other formulations. Below, there is an example of another version.

Problem 4.4. *An election is being held, and the voting for two candidates is made by pressing one of two buttons. How many ways are there for one of the candidates to get m votes and for the other to get n votes?*

Clearly, this problem is essentially a rewording of any of the previous ones. To vote for one of the candidates is to increase one of the components of a pair from the third version of the problem by 1. It is also similar to choosing the north or east direction, as in the original formulation of the problem.

Now, let us solve the problem in its original setting.

So let there be an $m \times n$ rectangle that is split into mn squares with sides of length 1 by lines, which are parallel to its sides. There are m squares along the horizontal side ("west-east" side) and n squares along the vertical side ("south-north" side). Starting from the south-western vertex (the point$A$ in Fig. 4.1) and moving along the lines and the sides of the rectangle, we have to reach the north-eastern vertex (the point$B$ in Fig. 4.1) of the rectangle in the shortest possible way. How many different eligible paths are there?

Each path is a sequence of $m+n$ elementary segments: m segments in "west-east" direction and n segments in "south-north" direction. The paths differ in the order of passing of these segments. Let E and N denote moves to the east and the north directions respectively. Then any sequence composed of m letters E and n letters N uniquely defines a certain path from A to B. Conversely, any path from A to B has a certain sequence of m letters E and n letters N corresponding to it. It is appropriate to call such a sequence a code of the corresponding path. Every code corresponds to a certain path and vice-versa. There is a bijection between the codes and paths. For instance, the paths highlighted in Fig. 4.1 have codes NEEEENNNNEEE containing m letters E and n letters N. In order to create such code, one has to choose m positions for the letter E out of available $m+n$ positions. The remaining n positions will be automatically filled by the letters N. There are C_{m+n}^{m} ways to choose m positions for the letter E. Therefore, we have the same number of different shortest paths from A to B. Obviously, the answer might be given by the symbol C_{m+n}^{n}, as well. For example, in a 7×5 rectangle depicted in Fig. 4.1, there are

$$C_{12}^{5} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$$

different shortest paths from A to B. Incredibly huge amount!

2. It will be convenient for us to use the coordinate model in this section. This means that we consider a rectangle on the coordinate plane, with its vertices being in the points with the following coordinates: $A(0;0)$, $C(0;n)$, $B(m;n)$, $D(m;0)$. The lines splitting the

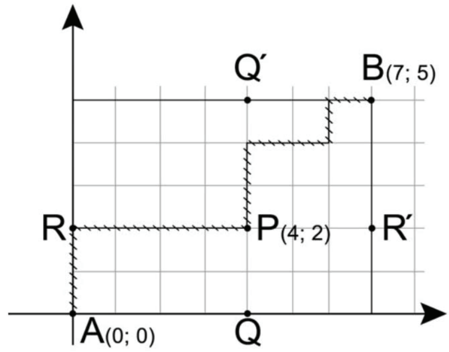

Figure 4.2. Shortest paths that connect two points.

rectangle lay on the coordinate lines $x = k\,(k = 1, 2, ..., m-1)$ and $y = s\ (s = 1, 2, ..., n-1)$. The sides of the rectangle also lay on the coordinate lines, namely: $x = 0$, $x = m$ and $y = 0$, $y = n$.

In such a coordinate model, the shortest path from A to B, which goes along the coordinate lines ($x = k$ and $y = s$, where k and s are integers), is defined by the following rule: the movement is always made in the direction of growth of abscissae ("eastbound"), or in the direction of growth of ordinates ("northbound"). As we have already learned, there are C^m_{m+n} (or, which is the same C^n_{m+n}) such paths in total.

Problem 4.5. *Let $P(u; v)$ be an integer point (u and v are integers) inside or on the bounds of the rectangle ACBD. This means that $0 \le u \le m$ and $0 \le v \le n$. Some of the shortest paths from the point $A(0; 0)$ to the point $B(m; n)$ go through the point (intersection) $P(u; v)$. How many such shortest paths exist?*

Solution. Denote by $Q(u; 0)$ and $R(0; v)$ the projections (orthogonal) of the point $P(u; v)$ on the axes of the abscissae and ordinates respectively. Any trajectory (the shortest path from A to B), which passes through the point P, can be split into two parts: a trajectory from A to P and a trajectory from P to B. The former is nothing else but the shortest path from A to P in the $u \times v$ rectangle $ARPQ$, and the latter is the shortest way from P to B in the rectangle $PQ'BR'$, where Q' and R' are orthogonal projections of the point P on the segments CB and BD. This is an $(m-u) \times (n-v)$ rectangle. Conversely, if we merge any trajectory from A to P in the rectangle $ARPQ$ with any trajectory from P to B in the rectangle $PQ'BR'$, then the resulting trajectory connects A and B and passes through the point$P$. This immediately yields that there are

$$C^u_{u+v} \cdot C^{m-u}_{m+n-u-v}$$

wanted trajectories.

Let us illustrate the above with an example for the exact values of m, n, u and v. Let $m = 7$, $n = 5$, $u = 4$, $v = 2$, that is we deal with the shortest paths that connect the points $A(0; 0)$ and $B(7; 5)$ and pass through the intersection $P(4; 2)$ (see Fig. 4.2).

The points A and P are the opposite points of the rectangle $ARPQ$. There are C^4_6 shortest paths from A to P. The points P and B are the opposite vertices of the rectangle $PQ'BR'$.

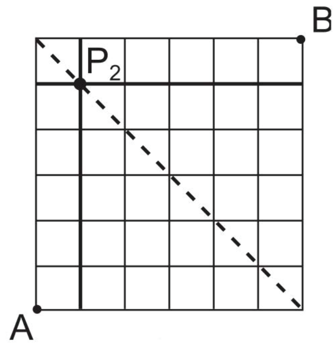

Figure 4.3. Two rectangles.

There are C_6^3 shortest paths from P to B. Hence, there are

$$C_6^4 \cdot C_6^3$$

shortest paths from A to B, which pass through the point P.

3. The shortest paths connecting the opposite vertices of a rectangle are a powerful source of interesting equalities involving the binomial coefficients. Let us begin with aesthetically perfect equality, which we have derived in another way in one of the previous chapters.

Consider an $n \times n$ square. Place it in the first quadrant of the coordinate plane, so that its vertices have the coordinates $A(0;0)$, $C(0;n)$, $B(n;n)$, $D(n;0)$. On the diagonal CD, there are $n+1$ integer points (points with integer coordinates), namely (we list them from the top left end of the diagonal to the bottom right: $C(0;n)$, $P_1(1;n-1)$, $P_2(2;n-2)$, $P_3(3;n-3)$, ..., $P_{n-1}(n-1;1)$, $D(n;0)$. Any trajectory connecting A and B passes through one (and only) of these points. Thus, we can split these trajectories into $n+1$ groups (according to the number of points on the diagonal): those that pass through the point C; those that pass through the point P_1; those that pass through the point P_2; and so on; finally, those that pass through the point D. Let us count the number of trajectories in each group. Consider the point $P_k\,(k;n-k)$. How many trajectories pass through it? The answer to this question is given by the problem from the previous section. There are C_n^k shortest paths from A to P_k. There is the same number of paths from P_k to B because P_k and B are the opposite vertices of the rectangle that is the same as the rectangle, which has A and P_k as its opposite vertices. These two rectangles are shown in Fig. 4.3 for $n=6$, $k=2$.

As we can merge any path from A to P_k with any path from P_k to B, then by virtue of the combinatorial rule of product, there are $(C_n^k)^2$ shortest paths from A to B passing through the point P_k. Therefore,

$(C_n^0)^2$ shortest paths pass through the point C;
$(C_n^1)^2$ shortest paths pass through the point P_1;
$(C_n^2)^2$ shortest paths pass through the point P_2;
etc.;
finally, $(C_n^n)^2$ shortest paths pass through the point D.

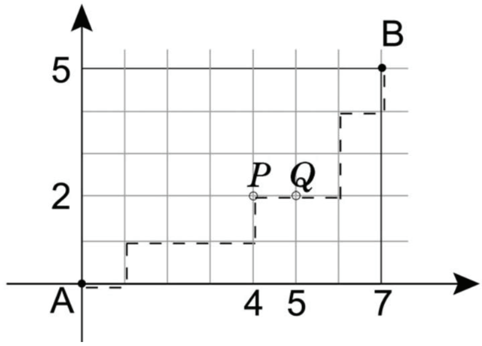

Figure 4.4. Shortest paths connecting points and passing the river over the bridge.

It comes from the result of the first section, that in an $n \times n$ square, there are C_{2n}^{n} shortest paths from A to B. Hence,

$$(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 + \ldots + (C_n^n)^2 = C_{2n}^n.$$

Brilliant, aesthetically perfect equality, which we have previously derived in a different context.

4. Bridge over a River. Again, we will deal with the shortest paths connecting the points $A(0;0)$ and $B(m;n)$ of the coordinate plane and passing along the lines $x = a$ $(a = 0, 1, \ldots, m)$ and $y = b$ $(b = 0, 1, \ldots, n)$. However, in this section, not all the shortest paths will be of interest. Imagine that the stripe bounded by the lines $x = s$ and $x = s+1$ $(0 \le s < m)$ is a river, and there is only one bridge over it, which connects $P(s;t)$ and $Q(s+1;t)$ $(0 \le t \le n)$. This bridge is the only way from one shore to another. How many passable shortest paths from A to B exist (that is, we consider only the paths that include PQ)?

The experience of the previous sections helps to answer this question with little effort. There are C_{s+t}^{s} shortest paths from A to P, and $C_{m+n-s-t-1}^{m-s-1}$ from Q to B. The shortest path from A to B is constructed with three segments: the shortest path from A to P, the unit segment (the bridge) PQ, and the shortest path from Q to B. In addition, the first and the third parts can be combined arbitrarily. Therefore, there are

$$C_{s+t}^{s} \cdot C_{m+n-s-t-1}^{m-s-1}$$

shortest paths from A to B involving the bridge PQ. For example, if $m = 7$, $n = 5$, $s = 4$, $t = 2$, then there are

$$C_6^4 \cdot C_5^2 = 150$$

shortest paths connecting the points A and B and passing the river over the bridge PQ. One of such paths is shown in Fig. 4.4.

5. Assume that inside the rectangle $ACBD$ with vertices in the points $A(0;0)$, $C(0;n)$, $B(m;n)$ and $D(m;0)$, there are $n+1$ bridges over the river with banks $x = s$ and $x = s+1$ $(0 \le s < m)$. The bridges are on the lines $y = t$ $(t = 0, 1, 2, .., n)$. Every shortest path from A to B inevitably passes through one (and only) of these bridges. We conclude that if

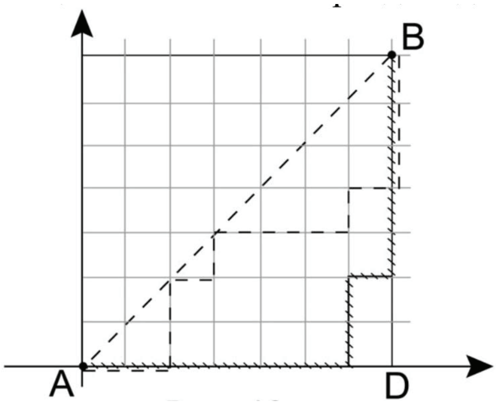

Figure 4.5. Subdiagonal paths in a square.

we count all paths going along each bridge and then sum up the results, then we will get the total amount of paths from A to B. As a result, we get a remarkable equality involving the binomial coefficients. Here it is:

$$C_s^s \cdot C_{m+n-s-1}^{m-s-1} + C_{s+1}^s \cdot C_{m+n-s-2}^{m-s-1} + C_{s+2}^s \cdot C_{m+n-s-3}^{m-s-1} +$$

$$+ \ldots + C_{s+t}^s \cdot C_{m+n-s-t-1}^{m-s-1} + \ldots + C_{s+n}^s \cdot C_{m-s-1}^{m-s-1} = C_{m+n}^m.$$

In particular, in the case $m = 7$, $n = 5$, $s = 4$, it turns into the following equality:

$$C_4^4 \cdot C_7^2 + C_5^4 \cdot C_6^2 + C_6^4 \cdot C_5^2 + C_7^4 \cdot C_4^2 + C_8^4 \cdot C_3^2 + C_9^4 \cdot C_2^2 = C_{12}^7.$$

6. In this section we again encounter the problem about the shortest paths in a square. Let $ACBD$ be a square with vertices in the points $A(0;0)$, $C(0;n)$, $B(n;n)$ and $D(n;0)$. The diagonal AB splits it into two triangles: the upper and the lower. Part of the shortest paths from A to B along the integer-valued coordinate lines lay in the lower triangle ABD (two such paths in a 7×7 square are shown in Fig. 4.5). We will call these paths subdiagonal throughout this chapter. Any of these paths do not get over the diagonal AB with any of its segments. They are allowed to have common points with this diagonal (the beginning, the end, and vertices of the polygonal chain). Other paths either cross the diagonal AB (have segments both below and over it), or completely lay above it (three of such paths are shown in Fig. 4.5). We are interested in the amount of paths of the second type, which are those that cross the diagonal AB or completely lay above it. There is a truly curious way to count them.

Let us draw the line $y = x + 1$ (see Fig. 4.7).

The shortest subdiagonal paths from A to B can have no common points with this line, as all of them are inside the triangle ABD. On the other hand, all other paths inevitably cross it or touch it. In other words, any other (not subdiagonal) path has at least one common point with this line. This is a very convenient feature, which enables us to distinguish subdiagonal paths from A to B from all other paths connecting these points. Below, we will show that thanking this property we can count the paths which go beyond the bounds of the triangle ABD, in a rather spectacular manner.

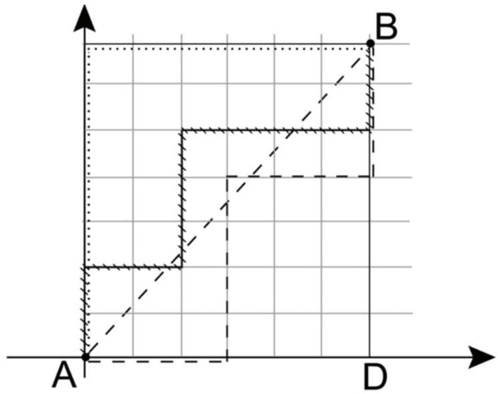

Figure 4.6. Nonsubdiagonal paths in a square.

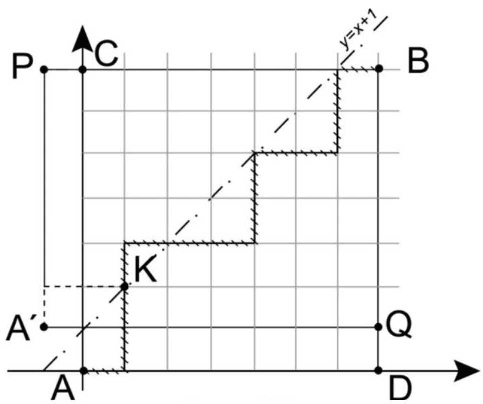

Figure 4.7. Symmetric image.

Let A' be asymmetrical (mirroring) image of the point A concerning the line $y = x + 1$. This point has the coordinates $A'(-1; 1)$. Along with the rectangle $ACBD$, consider the rectangle $A'PBQ$. This is an $(n+1)\times(n-1)$ rectangle (Fig. 4.6 depicts the case $n = 7$). Let us consider any shortest path from A to B, which goes beyond the bounds of the triangle ABD. As we already know, it necessarily has common points with the line $y = x + 1$. Let K be the first of such points on the way from A to B. It splits the polygonal chain, which is the path from A to B, into two parts. Let us call them as follows: the first part (from A to K) is the opening part, and the second part (from K to B) is the closing one. Mirroring the initial vertex of the path with respect to the line $y = x + 1$ and supplementing this symmetrical image with the closing part of the path, we get the shortest path from the point A' to the point B that is located in the rectangle $A'PBQ$. Applying this tricky method, we match every path from A to B, which crosses or touches the line $y = x + 1$, with another shortest way: from A' to B. Moreover, this correspondence is bijective: according to the introduced rule, every path from A' to B has one of the paths from A to B, which touch or cross the line $y = x + 1$, corresponding to it. This is the result of the fact that any path from A' to B

necessarily has common points (at least one) with the line $y = x+1$, as both its ends, the points A' and B, lay on opposite sides from this line. Moving along this path from its point of beginning A', we will inevitably meet some point of the line $y = x+1$. Let K be the first of such points. Replacing the initial part of this path, from the point A' to the point K, with its image with respect to the straight line $y = x+1$, we get the path from A to B, which has a common point with the line $y = x+1$.

The bijection established above completes the solution. It evidences that there are the same numbers of trajectories from A to B that go beyond the bounds of the triangle ABD, and of trajectories connecting the opposite sides A' and B of the rectangle $A'PBQ$. We have already learned the latter amount. As the size of rectangle is $(n+1)\times(n-1)$, the amount of wanted trajectories is C_{2n}^{n+1}.

7. ***The Catalan numbers.*** How many different subdiagonal trajectories are there in an $n \times n$ square? In order to avoid misunderstanding, let us rephrase the question more clearly. Let $ACBD$ be a square, where $A(0;0)$, $C(0;n)$, $B(n;n)$, $D(n;0)$ are its vertices. We are interested in paths from the vertex A to the vertex B constructed along the integer-valued coordinate lines. We consider only the shortest paths. All of them have length $2n$ (n steps north and n steps east). Some of these paths do not extend beyond the bounds of the triangle ABD. We call these paths subdiagonal, because they are located under the diagonal of the square $ACBD$.

How many such paths exist?

The answer to this question stems from the result of the previous section. There are C_{2n}^{n+1} paths that extend beyond the bounds of the triangle ABD. Other paths are subdiagonal. There are

$$C_{2n}^{n} - C_{2n}^{n+1} = \frac{(2n)!}{(n!)^2} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{1}{(n+1)}\cdot\frac{(2n)!}{(n!)^2} = \frac{1}{n+1}C_{2n}^{n}$$

of them.

As we can see, subdiagonal trajectories form $\frac{1}{n+1}$-th part of all trajectories.

The numbers $\frac{1}{n+1}C_{2n}^{n}$ are called the Catalan numbers. They are frequently encountered within combinatorial problems. Let us recall some of them.

In the chapter on bijection, we have talked about the close relationship between three important combinatorial problems: the problem about subdiagonal paths in an $n \times n$ square, the problem about triangulations of a convex n-gon and the problem of placing parentheses in a sum with n summands. We strongly recommend reviewing the corresponding part of this book, and here we will briefly remind the obtained results.

We used to denote the number of subdiagonal paths in an $n \times n$ square by $p(n)$. We have the direct formula for this amount now:

$$p(n) = \frac{1}{n+1}C_{2n}^{n}.$$

Triangulation of a convex polygon is the partition of this polygon into triangles by its non-intersecting (inside the polygon) diagonals. The question about the number of different triangulations of an n-gon (which is a polygon with n sides) is a typical combinatorial problem. Let us denote this amount with the symbol $t(n)$. Above, we have derived that

$$t(n) = p(n-2).$$

Having developed the direct formula for $p(n)$, it is straightforward to find the corresponding formula for $t(n)$:

$$t(n) = \frac{1}{n-1} C_{2(n-2)}^{n-2}.$$

Thus, the Catalan numbers appear in the problem about the triangulations of polygons, as well. Finally, recall the problem about the placement of parentheses in a sum of several summands. Explain it in the case of five summands:

$$a+b+c+d+e.$$

Assume that it is possible to add only two summands, hence, to compute the suggested sum, we need to group the summands in such a way that it turns into a chain of additions of two summands. This can be achieved by the appropriate placing of parentheses. For example, placing the parentheses as follows

$$((a+b)+((c+d)+e)),$$

we proceed with summing according to the chain

$$a+b=u,\ c+d=v,\ v+e=t,\ u+t=s$$

(s is the sought sum). Obviously, there is another option:

$$c+d=v,\ a+b=u,\ v+e=t,\ u+t=s.$$

Although parentheses do not dictate the order of actions unambiguously, they still introduce certain consistency to the procedure. Particularly, in our case, the additions denoted by the first and the third "+" signs should be performed before the one marked by the second "+" sign. Every pair of parentheses relates to an exact "+" sign and denotes the area of its responsibility. This type of mutual correspondence between the pairs of parentheses and the "+" signs is evidence that the parentheses are placed correctly.

From the combinatorial point of view, the following question is very interesting: how many different ways are there to place parentheses in a sum of n summands?

In the corresponding chapter, we have denoted this number by the symbol $d(n)$ and have proved that

$$d(n) = t(n+1).$$

Having discovered that

$$t_n = \frac{1}{n-1} C_{2(n-2)}^{n-2},$$

we can conclude that

$$d(n) = \frac{1}{n} C_{2(n-1)}^{n-1}.$$

We have seen above in this book that any path in an $n \times n$ square can be encoded by a sequence of n letters E (eats) and n letters N (north). Every (shortest) path between two opposite vertices of a square has one of these sequences corresponding to it (it can be considered to be the code of respective path. Conversely, every path in a square corresponds to a certain sequence. One may not stick to the letters E and N and may use any two

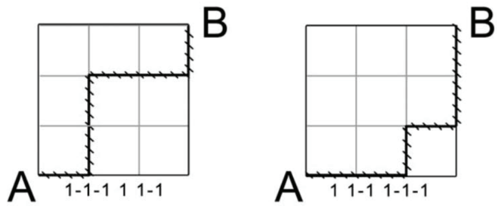

Figure 4.8. Sample codes of paths.

(different) symbols to encode the paths. For instance, replacing the letter N with the number "1" and the letter E with the number "-1", one gets an alternative encoding scheme, where the role of codes is played by the sequences of these two numbers of length $2n$ (that is, the sequences composed of $2n$ symbols each). Of course, an arbitrary sequence of the described type is the code of some path only if it contains the same amount of numbers "1" and "-1" (n of each). This pattern of a code can be replaced by another: a sequence is a code of some path in an $n \times n$ square if and only if the sum of its elements is 0. For example, the sequences $11-11-1-1$ and $1-1-111-1$ are codes of paths in a 3×3 square (these paths are shown in Fig. 4.8), and the sequences $1-1-1111$ or $1-11-1-1-1$ do not correspond to any shortest paths between the opposite vertices of this square.

Which is the characteristic property of the codes of subdiagonal paths? What distinguishes them from all other paths?

It is straightforward to answer these questions. The symbol "1" denotes the unit step (movement along the side of one unit square) to the right, and the symbol "-1" denotes the unit step upwards. In order to get to any subdiagonal intersection, one needs to take more steps to the right than upwards. In order to get to an intersection on the diagonal AB, the numbers of steps of two types have to be equal. Therefore, if we want a path from A to B to be subdiagonal (which is equivalent to this path passing through intersections on or below the diagonal), we need to ensure that at each point on our way from A to B we have not made more vertical steps than the horizontal ones. This feature of a subdiagonal path can be easily checked by its code. If we consecutively add the numbers 1 and -1 that form the code of a subdiagonal path, we get the sequence of non-negative sums. Alternatively, if a code defines a path that extends above the diagonal AB (that is, it passes through at least one intersection located above the diagonal), then there is at least one negative among the above sums.

It remains to accompany the above with a proper illustration. We provide several sample codes of paths for the case of a 4×4 square.

The code $1-111-11-1-1$ defines a subdiagonal path because the sequence of its sums mentioned above (it is appropriate to call these sums sequential partial sums) does not contain negative values. The sequence is: 1, 0, 1, 2, 1, 2, 1, 0. On the other hand, the code $1-11-1-111-1$ defines a path from A to B, part of which extends above the diagonal AB because the sequence of its partial sums 1, 0, 1, 0, -1, 0, 1, 0 contains a negative number.

As it has been shown above, there are $\frac{1}{n+1}C_{2n}^{n}$ (n-th term in the sequence of the Catalan numbers) subdiagonal ways in an $n \times n$ square. We are familiar now with another combi-

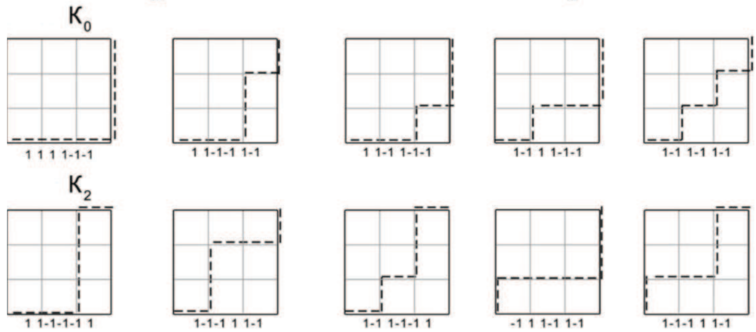

Figure 4.9. Classification of the shortest paths in a square. K_0,K_2.

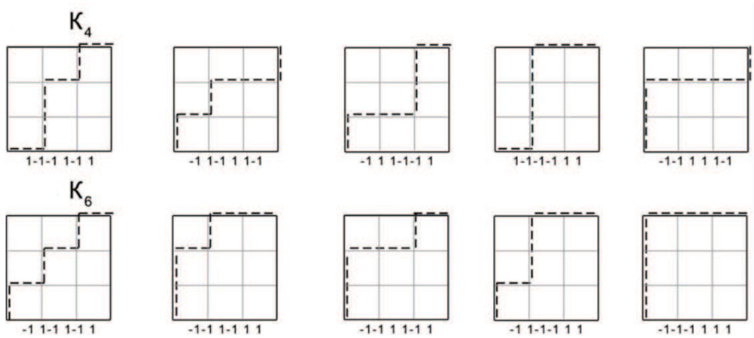

Figure 4.10. Classification of the shortest paths in a square. K_4, K_6.

natorial model for this number. It denotes the amount of sequences of n numbers "1" and n numbers "-1", which have non-zero partial sums.

8. ***One classification of the shortest paths in a square (the Chung-Feller theorem).*** Split all paths in an $n \times n$ square into $n+1$ classes:

the class K_0 contains those paths which have none of their line segments (of length 1) above the diagonal AB (i.e., subdiagonal paths);

the class K_2 contains those paths which have two of their line segments above the diagonal AB;

the class K_4 contains those paths which have four of their line segments above the diagonal AB;

In short, the class K_{2s} $(s = 0, 1, 2, \ldots, n)$ contains those paths which have $2s$ of their line segments above the diagonal AB.

In fact, none of the shortest paths can have an odd number of its line segments above the diagonal, because such paths should have equal amounts of vertical and horizontal line segments.

Fig. 4.9, Fig. 4.10 illustrate the above for the case of 3×3 square. All the shortest paths are split into 4 classes depending on the number of their unit line segments laying above the diagonal AB. Under each path, there is its arithmetical code.

As we can see, in a 3×3 square, all possible shortest paths fall into four groups of the same size (five paths in each) depending on the amount of line segments located above the diagonal: zero, two, four, or six.

An intriguing question is: does this property of shortest paths extend on any $n \times n$ square?

In 1949, well-known in mathematical world scientists K Chung and W. Feller found that the answer to the above question is positive. Actually, they were the ones who posed this question and answered it proving that all classes $K_0, K_2, K_4, \ldots, K_{2n}$ are equally sized.

We present one of the possible proofs of this interesting fact.

Let s be an integer from 0 to $n-1$ inclusive. Let us establish a bijection between the paths belonging to the class K_{2s} and the paths from $K_{2(s+1)}$. While doing so, we will consider not the paths themselves but their numeric codes ($2n$-sequences of numbers 1 and -1). First, determine the codes of the paths from the class K_{2s}. Clearly, the r-th line segment of any path lays above the diagonal if and only if one of the following two statements is true: either r-th partial sum of its code is negative, or it is zero but the previous one ($(r-1)$-th) the partial sum is negative. Thus, a path belongs to K_{2s} if and only if its code contains exactly $2s$ such partial sums.

Consider a code of arbitrary path in K_{2s}. We construct its partial sums until the first occurrence of the partial sum equal to 1 (such sum exists because it is given that $s < n$, which means that the path can not entirely evolve above the diagonal). Suppose that this partial sum has the number $p+1$. We proceed with the building of the partial sums up to the moment when we get the partial sum equal to zero (for the first time since the sum number 1). Again, this will inevitably happen as the sum of all numbers of the code (i.e., the partial sum number $2s$) is zero. Suppose that the sought sum has the number $p+q+2$. Thus, the code of any path from K_{2s} has the following form

$$\alpha_1 \ldots \alpha_p 1 \beta_1 \ldots \beta_q - 1 \gamma_1 \ldots \gamma_t, \tag{4.1}$$

where α_i, β_i and γ_i are 1 or -1. Evidently, $p+q+t = 2n-2$. Note that the numbers $\alpha_1, \ldots, \alpha_p$ can be absent. But if they are available, then $\alpha_1 = -1$, $\alpha_p = 1$, $\alpha_1 + \alpha_2 + \ldots + \alpha_p = 0$, p is even (otherwise, the sum $\alpha_1 + \alpha_2 + \ldots + \alpha_p$ could not be equal to zero). In addition, there is no positive among p partial sums; in particular, $p \le 2s$ (why?). The groups of numbers $\beta_1, \beta_2, \ldots, \beta_q$ and $\gamma_1, \gamma_2, \ldots, \gamma_q$ can be absent as well. However, if they are not, then q and t are even and $\beta_1 + \beta_2 + \ldots + \beta_q = 0$ and $\gamma_1 + \gamma_2 + \ldots + \gamma_t = 0$ (why?)

Transform code (4.1), placing the group of numbers $\alpha_1\alpha_2 \ldots \alpha_p 1$ right after the outlined number -1. We get the code

$$\beta_1\beta_2 \ldots \beta_q - 1 \alpha_1 \alpha_2 \ldots \alpha_p 1 \gamma_1 \ldots \gamma_t \tag{4.2}$$

of some other path. Let us ascertain that this path belongs to $K_{2(s+1)}$. None of the initial q partial sums of code (4.2) is negative because otherwise one of the sums of code (4.1) with number from $p+2$ to $p+q+1$ would be equal to zero, and the latter is just not true. The partial sums with numbers from $q+1$ to $p+q+1$, $p+1$ sums altogether, are negative (as $\beta_1 + \beta_2 + \ldots + \beta_q = 0$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_k \le 0$ for $k = 1, 2, \ldots, p$). Finally, the partial sum with the number $p+q+2$ equals to zero and the previous one is negative. All other partial sums (with numbers from $p+q+3$ to $2n$) of codes (4.1) and (4.2) are the same. Hence, the

path that corresponds to code (4.2) has two more line segments above the diagonal than the path defined by code (4.1). Therefore, the former path belongs to $K_{2(s+1)}$.

In code (4.2), the outlined numbers -1 and 1 also play a specific role: the partial sum that ends with the outlined summand -1 is the first of all those, which are equal to -1, and the partial sum ending with the outlined 1 is the first zero among all sums that follow it. This means that under a mapping, which transforms codes (4.1) into codes (4.2), different codes from K_{2s} have different images in $K_{2(s+1)}$. Indeed, if in two codes from K_{2s}

$$\alpha_1\alpha_2...\alpha_p 1\beta_1\beta_2...\beta_q - 1\gamma_1\gamma_2...\gamma_t \quad \textit{and} \quad \alpha'_1\alpha'_2...\alpha'_{p'}1\beta'_1\beta'_2...\beta'_{q'} - 1\gamma'_1\gamma'_2....\gamma'_{t'} \tag{4.3}$$

the outlined numbers 1 or -1 appear in different positions (that is, $p \neq p'$ or $p+q \neq p'+q'$), then their images also differ in the positions of the outlined numbers -1 or 1. Alternatively, if $p = p'$ and $q = q'$, then codes (4.3) have different numbers at least in one non-outlined position (in fact, at least in two positions; why?). It is obvious that in this case their images are also different.

Let us summarize. We established the rule under which the code of each path from K_{2s} can be transformed into the code of a path from $K_{2(s+1)}$. In addition, different codes of paths from K_{2s} transform into different codes of paths from $K_{2(s+1)}$. The conclusion is that the amount of paths in $K_{2(s+1)}$ is greater or equal to the amount of paths in K_{2s}:

$$|K_{2s}| \leq \left|K_{2(s+1)}\right|.$$

We emphasize that this conclusion concerns to any integer value s from 0 to $n-1$ inclusive. Thus, we have proved that

$$|K_0| \leq |K_2| \leq |K_4| \leq |K_6| \leq ... \leq \left|K_{2(n-1)}\right| \leq |K_{2n}|.$$

Now, concentrate on the first and the last of the above chain of inequalities. What is K_{2s}? This is the set of all subdiagonal paths. The symbol $|K_0|$ means the number of elements of the set K_{2s}, that is the number of subdiagonal paths. And what is A_{2n}? This is the set of all those paths that have $2n$ of their unit line segments above the diagonal. But a path in an $n\times n$ square has $2n$ line segments in total. Therefore, K_{2n} is the set of all non-subdiagonal paths. Thus, the symbol $|K_{2n}|$ denotes the amount of non-subdiagonal paths. The symmetry of the square $ACBD$ with respect to the diagonal AB establishes a bijection between subdiagonal and non-subdiagonal paths from A to B. Hence, $|K_0| = |K_{2n}|$. Take another look at the chain of inequalities derived above. The numbers $|K_0|$ and $|K_{2n}|$ are its first and last components. We have just realized that these numbers are equal. There is no other way for the remaining components but to join them in their equality. Therefore, we actually have the chain of equalities

$$|K_0| = |K_2| = |K_4| = |K_6| = ... = \left|K_{2(n-1)}\right| = |K_{2n}|.$$

The Chung-Feller theorem is proved.

It is worth noting that the obtained result is the new proof of the formula for the number of subdiagonal paths. Indeed, there are C_{2n}^n shortest paths connecting the opposite vertices of an $n\times n$ square. We have just discovered that these paths fall into $n+1$ equally sized classes depending on the number of their unit line segments that are located above the

diagonal. There can be 0, or 2, or 4, and so on up to $2n$ such line segments. Hence, every class contains $\frac{1}{n+1}C_{2n}^{n}$ paths.

9. The results of this section evidence that schemes involving shortest paths in a square serve as models for various combinatorial problems with absolutely different plots and scope. For instance, recall the problems of the placing of parentheses in a sum (or product) of several terms or the problems concerning the triangulation of a convex polygon. Let us add one more interesting problem to this list.

Imagine that two persons named A and B are the only participants in a pool championship. The one who wins n matches first is declared the champion. They are about to play until the champion is determined.

Question: how many different courses of the competition are possible?

First, let us clarify the setting. The report about the course of competition can be expressed in the form of a sequence of letters A and B, which denote the winner of one or another match. In addition, upon the end of the tournament such sequence should contain n letters? and less than n letters B or n letters B less than n letters A. Sequences of the first type (where the letter A dominates) evidence that player A has won, while sequences of the second type report player B's victory. It appears that we are required to find the number of these sequences, which serve as a report about the course of the competition.

For example, if $n = 3$ (the players agreed to play until one of them gets three wins), then there are 20 such sequences, namely:

AAA	BBB
$AABA$	$BBAB$
$AABBA$	$BBAAB$
$ABAA$	$BABB$
$ABABA$	$BABAB$
$ABBAA$	$BAABB$
$BAAA$	$ABBB$
$BAABA$	$ABBAB$
$BABAA$	$ABABB$
$BBAAA$	$AABBB$

The sequences in the left column evidence the victory of players. In particular, the first of them corresponds to the case when player A wins three times in a row in three opening matches. The second sequence tells that player A wins in the first, second, and fourth matches and loses the second one. The sequences of the right column describe the victories of player B. All of them can be obtained by switching the letters in the sequences of the left column.

Every sequence is probably the briefest report about the course of the competition and its winner. It provides an answer to any question concerning the tournament: how many matches were played, how many matches were won by the one who lost the entire tournament, who won the second match, etc.

How do these sequences relate to the shortest paths connecting the opposite vertices of a square? We will find the answer to this question with the help of the particular case of $n = 3$ presented above in this chapter. Compare the sequences arranged into two columns as above with the shortest paths between the points $A\,(0;0)$ and $B\,(3;3)$, stretching along

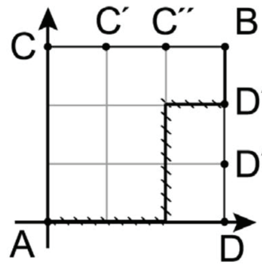

Figure 4.11. Paths and codes.

the coordinate lines $x = k$ and $y = s$ $(k, s = 0, 1, 2, 3)$. At a glance, it may appear that there is not much in common between the codes of courses of the competition and the codes of paths, except the both are constructed with the same letters. The codes of the shortest paths between A and B contain a fixed number of positions – six. There is the letter R (right) at three of them, and three other positions contain the letter U (up). Alternatively, one can use any other notation, for instance, the one introduced above – E (east) and N (north). As to the codes of courses of the competition, there is no predetermined length for them, and in particular, they never contain six positions. However, the connection between the objects (codes) of both types does exist. In order to identify and realize it, take a look at Fig. 4.11. For example, consider the code AABBA. Interpret it as a code of a certain movement along the coordinate lines from the point $A\,(0;0)$. Assume that the letter A means a step to the right and the letter B denotes an upward move. Where will the defined path end? Obviously, at the point D'' (the resulting path is highlighted in Fig. 4.11).

Let us take another code from the first column, say, BAAA. Again, consider it as a code of a polygonal chain that begins in the point $A\,(0;0)$. Then this chain will end in the point D'. The question is: if we decode the sequences of the first column as rules of movement on the coordinate plane from the starting point $A\,(0;0)$, then where will the journeys end? Clearly, in one of the points D, D' or D''. Why? Because every sequence of the first column contains three letters A and at most two letters B. Following the order of movement encoded in any sequence of the first column, we will make three steps in the direction of growth abscissa and at most 2 steps in the direction of growth of ordinate. That is why we will inevitably find ourselves in one of the integer points of the line segment BD, excluding the point $B\,(3;3)$. It is now straightforward to guess the relationship between the sequences of the first column and the codes of the paths from $A\,(0;0)$ to $B\,(3;3)$. The sequences are the initial fragments of the codes of those paths that end with (one or two) vertical unit line segments. In addition, these initial fragments are not taken randomly. Instead, we are restricted to those fragments that can be supplemented to the whole paths in a special way, namely, by the addition of some amount of the letters B. There is a diversity of trajectories of the path in the square $ACBD$ on its way to the side DB, but once it reaches it, the path will lead north to the letter B.

The above concerns the sequences from the second column as well. They are the initial fragments of the codes of those paths from A to B that end with (one or more) letters A. Geometrically, these are the paths the last line segment of which lay on the line CB.

Thus, we can say that all 20 sequences of the letters A and B are the cut codes of paths from $A\,(0;0)$ to $B\,(3;3)$, and the cutting is performed in such a way that the complete codes can be fully recovered from the remaining fragments. Below, there is the complete list of sequences responsible for the course of competitions (cut codes of paths), accompanied with the complete codes recovered from them:

AAA	$AAABBB$	BBB	$BBBAAA$
$AABA$	$AABABB$	$BBAB$	$BBABAA$
$AABBA$	$AABBAB$	$BBAAB$	$BBAABA$
$ABAA$	$ABAABB$	$BABB$	$BABBAA$
$ABABA$	$ABABAB$	$BABAB$	$BABABA$
$ABBAA$	$ABBAAB$	$BAABB$	$BAABBA$
$BAAA$	$BAAABB$	$ABBB$	$ABBBAA$
$BAABA$	$BAABAB$	$ABBAB$	$ABBABA$
$BABAA$	$BABAAB$	$ABABB$	$ABABBA$
$BBAAA$	$BBAAAB$	$AABBB$	$AABBBA$

Conclusion. In a certain sense, there is a "natural" bijection between the codes of courses of competitions and the codes of shortest paths in a 3×3 square, which evidences that the amounts of objects of both types are equal to C_6^3, as this is the amount of the objects of the latter type. Hence, if two players participate in a game with no draws until three wins of one of them, then there are C_6^3 possible courses of such game.

Needless to say, the above considerations can be easily adjusted to the general case, where the game continues until the n victories of one of the players. The only essential change concerns the lengths of the codes of paths and the codes of courses of competition. The main ingredient, which is the procedure of establishing a bijection between the codes of both types (codes of courses of competition up to n victories of one of the players and codes of paths connecting the points $A\,(0;0)$ and $B\,(n;n)$), does not change. Therefore, we conclude: if two players participate in a game with no draws until three wins of one of them, then there are C_{2n}^n possible courses of such game.

10. ***Graphs of functions located on intersecting lines.***

Draw six straight lines on the coordinate plane (see Fig. 4.12)

$$y=x+1,\ y=x,\ y=x-1,\ y=-x+1,\ y=-x,\ y=-x-1.$$

Points of intersection of three first of them with three others form a square $PQTS$, which is split by the middle lines into 4 smaller squares. Consider these six lines with no regard to the coordinate axes. In the combined graph of the above lines, we will further call the line L.

We have to answer the following question:

How many different functions possess three following properties:

a) a function is defined on R (the set of all real numbers);

b) it is continuous (there are no discontinuities on its graph);

c) its graph is part of the line L?

Recall that an arbitrary line on the coordinate plane can not be a graph of a function unless any vertical straight line (the line $x=c$) crosses have at most one point of intersection

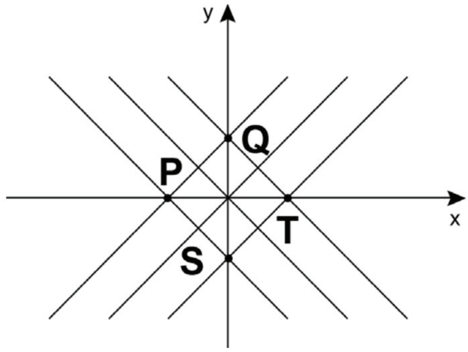

Figure 4.12. Graphs of functions located on intersecting lines.

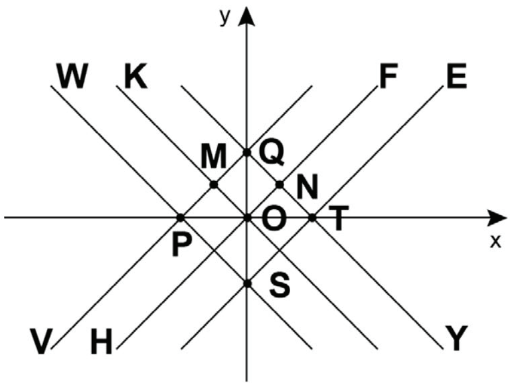

Figure 4.13. Polygonal chains.

with it. If a function is defined on the set of all real numbers (on R), then each vertical line intersects with its graph in one point. In view of properties b) and c) how does this fact affect the structure of a graph in our case? Which part of L it can be, and which it can not? The graph of every our function is a polygonal chain (or a straight line) which "comes" from the north or northwest, travels inside the square $PQTS$, and vanishes in the north- or south-eastern direction. Imagine a point, which appears on the left ray of this polygonal chain and moves to the right. Then the vector of the velocity of this point is always directed to the northeast or southeast, changing its direction in the vertices of the polygonal chain.

All six graphs of initial functions are among the wanted ones. All other graphs are polygonal chains composed of two (left and right) rays and, possibly, several line segments from the square $PQTS$. Here are some of these polygonal chains (see Fig. 4.13): $KMQNF, KMQTE,\ KONJ, HNTE,\ HNJ, VPSTJ, VMOF,\ VMONTE, VQTE$.

So how many such polygonal chains are there in total? Arranging the process of search, finding enough patience and concentration, we could eventually derive the number of graphs in this case, where the line L is constructed with six lines. However, such a result would not have much value, because solving this type of special case, it is desirable to bear in mind their natural generalizations.

If one takes a careful look at Fig. 4.13, concentrating on those zigzags inside the square $PQTS$ which compose part of wanted polygonal chains, then it will definitely appear that the problem about graphs is somehow connected with the problem of the shortest paths between the opposite vertices in a square, which we have dealt with above in this section. If this is the case, if there is truly a bijection or some correspondence close to it between the graphs and the paths in a square, then this is not the square $PQTS$, as a substantial part of graphs does not pass through the points P and Q at once or even through one of them. For example, the graphs $WPMOF$ or $WPQNF$ contain the point P but not T, and the graphs KOF, $KMQNF$ or HF do not contain any of the two. In order to simplify calculations, we have to strengthen our feeling of the alleged connection between the graphs of functions in L and the shortest paths in a square. To detect patterns, we have to consider several special cases with different numbers of lines and different sizes of squares. Let us investigate the following options for the lines: 1) two lines of opposite direction (for example, $y = x$ and $y = -x$); 2) two lines with the one direction (parallel to the line $y = x$) and two lines with the other (parallel to the line $y = -x$); 3) two triplets of lines from the original problem. Having thoroughly counted the graphs of everywhere defined continuous functions that form the part of the line (combined graph) L, we get the results, which are best presented in the form of the table 4.1:

Table 4.1. Number of graphs.

Number of lines	1 and 1	2 and 2	3 and 3
Number of graphs	4	18	68

Below, there is a table of numbers C_{2n}^{n} for small values of n. As we already know, these numbers express the amounts of the shortest paths between the opposite vertices of an $n \times n$ square:

Table 4.2. Number C_{2n}^{n}.

n	1	2	3	4
C_{2n}^{n}	2	6	20	70

If we stopped building the second table after three initial values of n, then almost certainly we would have missed the similarities between both tables and overlooked a brilliant discovery. However, now, it is hard not to notice that ignoring the first column of the second table, we have a very simple relation between the numbers 4, 18, and 68 in the second row of the first table and the numbers 6, 20, and 70 in the second row of the second table: $6-4=2$, $20-18=2$ and $70-68=2$. This observation is the basis for a hypothesis:

The combined graph of n lines parallel to the line $y = x$, and n lines parallel to the line $y = -x$ can contain

$$C_{2(n+1)}^{n+1} - 2$$

graphs of everywhere defined continuous functions.

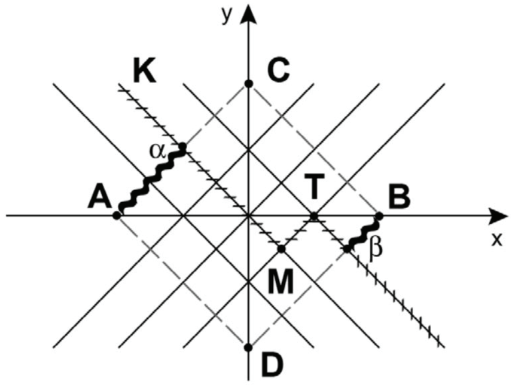

Figure 4.14. Bijection between the graphs of functions and the shortest paths.

As we can see, when the combined graph L is constructed of $2n$ lines (n of each of two directions), then the wanted polygonal chains which form the part of this graph, are related with the paths in an $(n+1)\times(n+1)$ square, not $n\times n$. At least, exactly this relation is observed between the above tables.

It remains to prove this hypothesis. First, we need to decide how to fit an $(n+1)\times(n+1)$ square in the combined graph L of $2n$ lines. It appears that this square has to cover the square $PQTS$, which is an $(n-1)\times(n-1)$ square.

Proceed with this assumption (Fig. 4.14 shows the corresponding configuration for the case $n=3$) and attempt to establish a bijection between the graphs of functions and the shortest paths connecting the points A and B (excluding two of these paths). For instance, let us take the graph $KMTU$. It crosses the bounds of the square $ACBD$ in two points. Denote them by the Greek letters α (the one that lays on the side adjacent to the vertex A) and β (the one that lays on the side adjacent to the vertex B). Replacing the ray αK with the line segment $A\alpha$ and the ray βU with the line segment βB on the graph, we get a trajectory leading from A to B. Conversely, when such trajectory is available (except for its two line segments laying on the sides of the square $ACBD$, such trajectory consists of segments of the lines of the graph L), then extending its second (from A) and penultimate segments beyond the bounds of the square $ACBD$ to the left and to the right respectively, we get an infinite polygonal chain that belongs to the graph L and is a graph of some function. Following the above rule, the only trajectory from A to B that can not be transformed into a polygonal chain of the graph L is the one that does not have segments inside (not on the bounds of) the square $ACBD$. There are two such trajectories: ACB and ADB. By our rule, a bijection is established between other trajectories and those polygonal chains of the graph L which are graphs of functions. Hence, the amount of graphs is equal to the amount of all trajectories from A to B minus 2, that is

$$C_{2(n+1)}^{n+1}-2.$$

In particular, for $n=3$ this formula gives 68, because

$$C_{2\cdot 4}^{4}=C_{8}^{4}=\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}=70.$$

Problems

Problem 4.6. *How many shortest paths connecting the points A and B on the coordinate plane, follow along integer-valued coordinate lines $x = a$ and $y = b$ (a, b are arbitrary numbers), where A and B have the following coordinates: 1) $A\,(-2;3)$, $B\,(1;-1)$; 2) $A\,(5;0)$, $B\,(-3;1)$; 3) $A\,(-4;5)$, $B(1;2)$; 4) $A\,(2;-4)$, $B\,(0;1)$? In each case, find the length $d\,(A,B)$ of the shortest path from A to B.*

Answer. 1) C_7^3, $d\,(A,B) = 7$; 2) C_9^1, $d\,(A,B) = 9$; 3) C_9^3, $d(A,B) = 8$; 4) C_7^2, $d\,(A,B) = 7$.

Problem 4.7. *How many shortest paths along integer-valued coordinate lines, which connect the points A and B, contain the point F, where the coordinates of the points are: $A(2;-4)$, $B(-2;3)$, $F(5;1)$?*

Answer. $C_8^3 \cdot C_9^2$.

Problem 4.8. *Let A, B, C and D be given points. How many shortest paths along integer-valued coordinate lines allow a round-trip by the scheme $A \to B \to C \to D \to A$, where the coordinates are: a) $A\,(3;0)$, $B\,(0;3)$, $C\,(-3;0)$, $D\,(0;-3)$; b) $A\,(5;2)$, $B\,(0;4)$, $C\,(0;7)$, $D\,(-2;1)$.*

Answer. a) $(C_6^3)^4$; b) $C_7^2 \cdot C_8^2 \cdot C_8^1$.

Problem 4.9. *Departing from the point $A\,(1;-2)$, one needs to get to the point $B\,(4;3)$ via the shortest path along integer-valued coordinate lines, which has at least one common point with the line $x = -1$. How many ways are there to complete this trip (how many different paths are there)?*

Answer. C_{12}^5.

Hint. Let A' be a symmetrical reflection of the point A w.r.t. the line $x = -1$ (What are the coordinates of A'?). Prove that the sought amount of paths equals the number of shortest paths between the points A' and B.

Problem 4.10. *Departing from the point $A\,(3;2)$, one needs to get to the point $B\,(6;6)$ using the shortest path along integer-valued coordinate lines, which includes at least one point of each of the lines $x = 1$ and $x = 8$. How many ways are there to complete this trip (how many different paths are there that satisfy the above conditions)?*

Answer. C_{15}^4.

Hint. Let A' be a symmetrical reflection of the point A w.r.t. the line $x = 1$, and A'' be a symmetrical reflection of the point A w.r.t. the line $x = 8$. Determine the coordinates of the points A' and A''. Establish a bijection between the paths from the statement of the problem and the shortest paths from A'' and B, which evolve along integer-valued coordinate lines.

Remark. The central idea of this problem (utilization of axial symmetries) reminds one brilliant classical construction problem:

Let A and B be points in the stripe bounded by two parallel lines l_1 and l_2. What should be the initial direction of a ray of light that starts at the point A, then reflects in turn from the lines l_1 and l_2 (once from each) and finally gets to the point B? Using a straightedge and compass, construct the trajectory of this ray. Recall that Recall that the angle of incidence equals the angle of reflection (see Fig. 4.15).

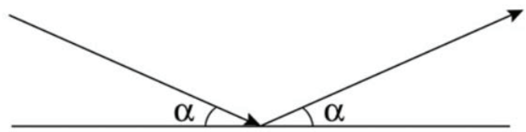

Figure 4.15. The angle of incidence equals to the angle of reflection.

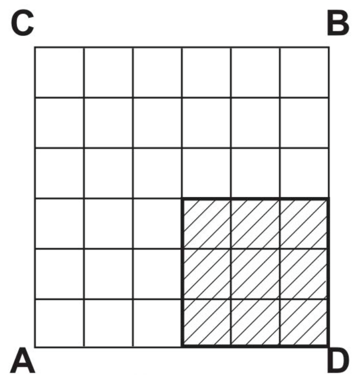

Figure 4.16. Road network and a restricted area.

Problem 4.11. *Solve the previous problem in the "general setting", assuming the points A and B have the coordinates: $A(u;v)$, $B(s;t)$, and the given lines are defined by equations: $x=a$ and $x=b$. In addition, $a<u<b$, $a<s<b$, which means that the points A and B are located between the lines.*

Answer. $C^{|v-t|}_{|u+2a-2b-s|+|v-t|}$.

Problem 4.12. *In Fig. 4.16, there is a 6×6 square ACBD with a familiar road network and a restricted area (hatched square). It is allowed to move along the bounds of the restricted area. How many shortest paths from A to B are there?*

Answer. $2+(C^1_6)^2+(C^2_6)^2+(C^3_6)^2$.

Problem 4.13. *1. How many shortest paths from $A(0;0)$ to $B(n;n)$ along integer-valued coordinate lines do not extend beyond the bounds of the area between the lines $y=x-1$ and $y=x+1$?*

2. How many of these paths:

a_0) share no points with the line $y=x+1$?

a_1) share one point with the line $y=x+1$?

a_2) share two points with the line $y=x+1$?

...

a_k) share k points with the line $y=x+1 \quad (0\le k\le n)$?

a_n) share n points with the line $y=x+1$?

3. What equality involving binomial coefficients, can be obtained by the comparison of the answers to questions 1) and 2)?

<u>Answer.</u> *1)* 2^n*; 2*a_k*)* C_n^k*; 3)* $C_n^0+C_n^1+C_n^2+\ldots+C_n^{n-1}+C_n^n=2^n$.

Problem 4.14. *How many shortest paths from* $A(0;0)$ *to* $B(n;n)$ *along integer-valued coordinate lines do not extend beyond the bounds of the area between the lines* $y=x$ *and* $y=x+2$*?*

<u>Answer.</u> 2^{n-1}.

Problem 4.15. *Consider a square ACBD with vertices in the points* $A(0;0)$, $C(0;8)$, $B(8;8)$ *and* $D(8;0)$. *We are going to be interested in the shortest paths from A to B that go along integer-valued coordinate lines. Every shortest path should include one (and only one) integer point (point with integer coordinates) of the line* $y=-x+7$. *Produce the list of coordinates of all integer points of this line that are located inside or on the bounds of the square ACBD, ordered ascendingly by abscissas. Count the shortest paths from A to B that pass through each of these points and express your answer with the binomial coefficients. What equality involving binomial coefficients can be derived based on obtained results?*

<u>Answer.</u> $C_7^0C_9^8+C_7^1C_9^7+C_7^2C_9^6+C_7^3C_9^5+C_7^4C_9^4+C_7^5C_9^3+C_7^6C_9^2+C_7^7C_9^1=C_{16}^8$.

Problem 4.16. *In the previous problem, replace the* 8×8 *square with the* $n\times n$ *square, and the line* $y=-x+7$ *with the line* $y=-x+(n-1)$. *Once again, find all integer points of the latter line, which are passed with the shortest paths from A to B (as above, list them in ascending order by their abscissa. Name three initial and three last points in the list). What equality involving binomial coefficients can be derived based on obtained results this time?*

<u>Answer.</u> $C_{n-1}^0C_{n+1}^n+C_{n-1}^1C_{n+1}^{n-1}+C_{n-1}^2C_{n+1}^{n-2}+\ldots+C_{n-1}^{n-3}C_{n+1}^3+C_{n-1}^{n-2}C_{n+1}^2+C_{n-1}^{n-1}C_{n+1}^1=C_{2n}^n$.

Problem 4.17. *What equality involving binomial coefficients can be defined, when the line* $y=-x+(n-1)$ *is replaced with the line* $y=-x+(n-2)$*?*

<u>Answer.</u> $C_{n-2}^0C_{n+2}^n+C_{n-2}^1C_{n+2}^{n-1}+C_{n-2}^2C_{n+2}^{n-2}+\ldots+C_{n-2}^{n-4}C_{n+2}^4+C_{n-2}^{n-3}C_{n+2}^3+C_{n-2}^{n-2}C_{n+2}^2=C_{2n}^n$.

Problem 4.18. *Generalize the equalities obtained in two previous problems. To this end, consider an arbitrary line* $y=-x+k$ $(0<k<n)$ *instead of the exact lines* $y=-x+(n-1)$ *and* $y=-x+(n-2)$. *Classify all possible shortest paths between A and B depending on the integer point of the above line, which they pass through. Determine the number of paths in each group and outline the related equality. In particular, write down this equality in the special cases of* $k=1$ *and* $k=2$.

<u>Answer.</u> $C_k^0C_{2n-k}^n+C_k^1C_{2n-k}^{n-1}+C_k^2C_{2n-k}^{n-2}+\ldots+C_k^{k-2}C_{2n-k}^{n-k+2}+C_k^{k-1}C_{2n-k}^{n-k+1}+C_k^kC_{2n-k}^{n-k}=C_{2n}^n$.

For $k=1$*:* $C_{2n-1}^n+C_{2n-1}^{n-1}=C_{2n}^n$.

For $k=2$*:* $C_2^0C_{2n-2}^n+C_2^1C_{2n-2}^{n-1}+C_2^2+C_{2n-2}^{n-2}=C_{2n}^n$.

Problem 4.19. *This is another problem on the shortest paths between the points* $A(0;0)$ *and* $B(n;n)$, *which lay on integer-valued coordinate lines. In the section about the Catalan numbers, it has been determined that there are* $\frac{1}{n+1}C_{2n}^n$ *shortest paths from* $A(0;0)$ *to* $B(n;n)$, *which do not have common points with the line* $y=x+1$. *The proof of this result has been very instructive in the way it has exploited the principle of equality. Applying similar classification, determine the number of the shortest paths connecting* $A(0;0)$ *and* $B(n;n)$ *that do not have common points with.*

<u>Answer.</u> $C_{2n}^n-C_{2n}^{n-2}$.

Hint. Let us ascertain that the number of shortest paths between the points $A\,(0;0)$ and $B\,(n;n)$, which have common points with the line $y = x+2$, is the same as the number of shortest paths between the points $A'\,(-2;2)$ and $B\,(n;n)$ (A' is the reflection of the point A w.r.t. the line $y = x+2$).

Problem 4.20. *How many shortest paths from $A(0;0)$ to $B(n;n)$ on integer-valued coordinate lines have common points with the line $y = x+1$ but do not have common points with the line $y = x+2$?*

Answer. $C_{2n}^{n-1} - C_{2n}^{n-2}$.

Problem 4.21. *How many shortest paths from $A\,(0;0)$ to $B\,(n;n)$ on integer-valued coordinate lines have common points with the line $y = x+k$ $(1 \le k \le n)$?*

Answer. $C_{2n}^{n} - C_{2n}^{n-k}$.

Problem 4.22. *How many shortest paths from $A(0;0)$ to $B(n;n)$ through integer-valued coordinate lines pass through (completely or partially) the inner points of the hexagon created by the lines $y = x+2$, $y = x-2$, $y = 0$, $x = 0$, $y = n$ and $x = n$?*

Answer. $C_{2n}^{n} - \frac{2}{n-1} C_{2(n-2)}^{n-2}$.

Problem 4.23. *(Classification of paths in a square by the number of line segments). Every shortest path from $A\,(0;0)$ to $B\,(5;5)$ that goes along integer-valued coordinate lines, is a polygonal chain consisting of several lines segments. It is straightforward to verify that (the shortest) polygonal chains connecting the points A and B can comprise 2 to 10 line segments. Thus, the can be split into 9 groups by this property (the number of line segments). Determine the number of paths in each group. Construct a table of two rows (or columns); the first row of this table should contain numbers from 2 to 9 that express the number of line segments of the polygonal chain and the second row shows the amount of corresponding polygonal chains. Every wanted number can be expressed as a product of two binomial coefficients. First, express the numbers in this format, and then find their exact values that should stand in the second row of the table.*

Having completed the above tasks, you will be able to construct a curious equality involving the binomial coefficients: in one of its sides is the sum of numbers, each of which expresses the number of paths that consist of a certain amount of line segments, and on the other side is the number C_{10}^{5}, which expresses the total amount of the shortest paths between the points A and B. What is this equality?

Answer. *The numbers of paths constructed of two, three, four, five, etc., up to ten segments are respectively*

$$2C_4^0C_4^0,\ 2C_4^1C_4^0,\ 2C_4^1C_4^1,\ 2C_4^2C_4^1,\ 2C_4^2C_4^2,\ 2C_4^3C_4^2,\ 2C_4^3C_4^3,\ 2C_4^4C_4^3,\ 2C_4^4C_4^4.$$

The sum of these numbers equals to the total amount of shortest paths connecting the points $A\,(0;0)$ and $B\,(5;5)$, hence,

$$2(C_4^0C_4^0 + C_4^1C_4^0 + C_4^1C_4^1 + C_4^2C_4^1 + C_4^2C_4^2 + C_4^3C_4^2 + C_4^3C_4^3 + C_4^4C_4^3 + C_4^4C_4^4) = C_{10}^{5}.$$

Hint. Every shortest polygonal chain connecting the points A and B consists of horizontal and vertical line segments that alternate with each other. In particular, this means that a chain can be constructed of equal amounts of segments of two types, or these amounts

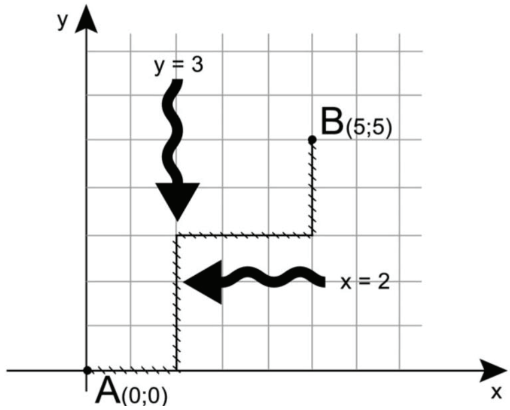

Figure 4.17. Classification of paths in a square by the number of line segments.

may differ by 1. Furthermore, notice that if l is one of those polygonal chains that begin with the horizontal segment (if departing from A), then its reflection l' from the line $y = x$ (the straight line AB) is the same polygonal chain except for its horizontal and vertical segments are replaced. As l is the reflection of l' if l' is the reflection of l, then creating the reflections of all the shortest polygonal chains connecting A and B, we establish a bijective correspondence between the chains which begin with horizontal segment and the chains beginning with vertical segment. This correspondence has an important additional feature. Two polygonal chains that correspond to each other are geometrically identical (as they overlay each other upon reflection). In particular, they consist of equal amounts of line segments. Thus, when it is necessary to determine the number of polygonal chains composed of a given amount of segments, we can act as follows: first, find the number of such polygonal chains among those that begins with a horizontal segment and then double up the result.

So now we have restricted ourselves to the chains that begin with horizontal segment only. How can we count the amount of those of them that consist of, say, four segments? Contemplate as follows. As there is an even amount of segments, and the first of them is horizontal (as agreed), then the last one (the fourth) should be vertical. Hence, the first segment lays on the line $y = 0$ (on the side AD of the square $ACBD$), and the last one is on the line $x = 5$ (on the side BD of the square $ACBD$). This means that the second segment (and it is necessarily vertical) lays on one of the lines $x = k$ $(k = 1, 2, 3, 4)$, and the third is on one of the lines $y = s$ $(s = 1, 2, 3, 4)$. It remains to note that any valid (one of the above four) values of k along with an arbitrary valid value of s define some polygonal chain with four line segments. It appears that such a chain is defined by the code $< \alpha; \beta >$ that consists of two numbers, which can attain values $1, 2, 3, 4$. The first number denotes the line to which the inner (not the last one) vertical segment belongs, and the second points out the line that contains the inner (not the first one) horizontal segment. For example, the code $(2; 3)$ defines the chain shown in c A bijection is established between the codes and four-segment polygonal chains. Therefore, the number of chains is the same as the number of codes, which is $4 \cdot 4(C_4^1 \cdot C_4^1)$

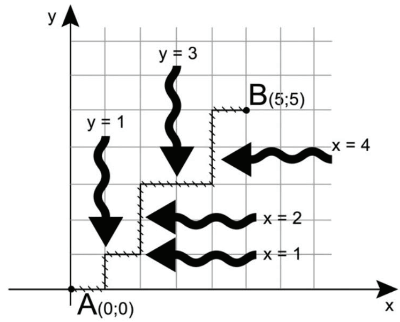

Figure 4.18. Bijection between codes and four-segment polygonal chains.

Let us count the polygonal chains, which begin with horizontal segment and consist of 7 segments. Such polygonal chains are constructed of 3 vertical and 4 horizontal segments. In particular, in addition to the first line segment the last one is also horizontal. Thus, there are 3 vertical and 3 horizontal inner segments. Vertical segments belong to three out of four lines $x = k$ $(k = 1, 2, 3, 4)$, and horizontal segments lay on two of four possible lines $y = s$ $(s = 1, 2, 3, 4)$. In order to choose one of seven-segment polygonal chains, one needs to specify the lines $x = k$ and $y = s$, to which its vertical and horizontal inner segments belong. Thus, a polygonal chain is defined by the code $< k_1, k_2, k_3; s_1, s_2 >$ composed of three different numbers k_i and two different numbers s_i, where $\{k_1, k_2, k_3\}$ and $\{s_1, s_2\}$ are subsets of the set $\{1, 2, 3, 4\}$. There are C_4^3 and C_4^2 subsets of the first and the second type respectively. Therefore, there are $C_4^3 \cdot C_4^2$ codes in total. And this is the amount of seven-element polygonal chains which begin with horizontal segment, because there is a bijection between the codes and the chains. In Fig. 4.18, the seven-segment chain with the code $< 1, 2, 4; 1, 3 >$ is shown.

Problem 4.24. *(Generalization of the previous problem). Consider the shortest paths between the points $A\,(0; 0)$ and $B\,(n; n)$ composed of the segments of lines $x = k\,(k = 0, 1, 2, ..., n)$ and $y = s$ $(s = 0, 1, 2, ..., n)$. Every such path is a polygonal chain. It may comprise two (there are two such chains) to $2n$ (two chains as well) line segments. To get from A to B along a polygonal chain composed of $2n$ line segments, one needs to change the direction of movement at each intersection. Let us split all possible chains (paths from A to B) into classes according to the number of line segments composing them. It is required to determine the power (number of elements) of each class. Basing on the experience of the previous problem, give answers to the following questions:*

1. *How many polygonal chains consist of 3 line segments? (How many paths from A to B have 2 turns?)*

2. *How many polygonal chains consist of 4 line segments? (How many paths from A to B have 3 turns?)*

3. *How many polygonal chains consist of 5 line segments? (How many paths from A to B have 4 turns?)*

4. *How many polygonal chains consist of 6 line segments? (How many paths from A to B have 5 turns?)*

5. *Let m be an integer from the interval* $[1,n]$, *i.e.,* $1 \le m \le n$.

a) How many polygonal chains consist of $2m-1$ *line segments?*

b) How many polygonal chains consist of $2m$ *line segments?*

Having determined the numbers that express the amount of polygonal chains in each class, construct equality for the binomial coefficients, which generalizes the equality from the previous problem.

Answer.
1) $C^1_{n-1}\cdot C^0_{n-1}$;
2) $C^1_{n-1}\cdot C^1_{n-1}$;
3) $C^2_{n-1}C^1_{n-1}$;
4) $C^2_{n-1}\cdot C^2_{n-2}$;
5) a) $C^{m-1}_{n-1}\cdot C^{m-2}_{n-1}$;
b) $C^{m-1}_{n-1}\cdot C^{m-1}_{n-1}$; $\quad 2\cdot(C^0_{n-1}\cdot C^0_{n-1}+C^1_{n-1}\cdot C^0_{n-1}+C^1_{n-1}\cdot C^1_{n-1}+C^2_{n-1}\cdot C^1_{n-1}+C^2_{n-1}\cdot C^2_{n-1}+ \ldots + C^{n-2}_{n-1}\cdot C^{n-2}_{n-1}+C^{n-1}_{n-1}\cdot C^{n-2}_{n-1}+C^{n-1}_{n-1}\cdot C^{n-1}_{n-1}) = C^n_{2n}$.

Problem 4.25. *Order the shortest paths connecting the points* $A(0,0$ *and* $B(10;4)$, *which are composed of segments of integer-valued coordinate lines, by a number of line segments. How many line segments can such a path be composed of? Construct the following table: in the first row, there should be numbers of line segments, and in the second row, the amounts of corresponding paths.*

Answer. There are 2 paths of two line segments, $(C^1_9C^0_3+C^1_3C^0_9)$ *paths of 3 segments,* $2C^1_9C^1_3$ *paths of 4 segments,* $(C^2_9C^1_3+C^2_3C^1_9)$ *paths of 5 segments,* $2C^2_9C^2_3$ *paths of 6 segments,* $(C^3_9C^2_3+C^3_3C^2_9)$ *paths of 7 segments,* $2C^3_9C^3_3$ *paths of 8 segments,* $C^4_9C^3_3$ *paths of 9 segments. There are no other paths. The sum of the above numbers is* C^4_{14}.

Problem 4.26. *Consider the shortest polygonal chains that connect the points* $A\,(0;0)$ *and* $B\,(n;m)$ *and evolve along integer-valued coordinate lines. The latter point has its abscissa greater than its ordinate (and both coordinates are integer). What the maximum number of line segments can such a polygonal chain be composed of? How many polygonal chains contain the maximum possible number of segments?*

Answer. $2m+1$; C^{m-1}_{n-1}

Problem 4.27. *Let L be a combined graph of n lines parallel to the line* $y=x$ *and n lines parallel to the line* $y=-x$. *At the end of the theoretical part of this chapter, it has been determined that there exist* $C^{n+1}_{2n+2}-2$ *polygonal chains that have all of the following features:*

a) any vertical line (line $x=a$*) crosses it in one point;*

b) it is a part of the combined graph L.

The above set of chains includes the lines that compose L because they satisfy the conditions a) and b). They may be considered to be polygonal chains that consist of one line segment only, and thus have no vertices. How many vertices can other polygonal chains have? And how many different polygonal chains have given amount of vertices? In particular:

1. *How many polygonal chains have one vertex?*
2. *How many polygonal chains have two vertices?*
3. *How many polygonal chains have three vertices?*
4. *How many polygonal chains have $2k-1$ vertices $(1 \le k \le n)$?*
5. *How many polygonal chains have $2k-1$ vertices $(1 \le k \le n)$?*

Answer. *1)* $2(C_n^1)^2$; *2)* $2C_n^2C_n^1$; *3)* $2(C_n^2)^2$; *4)* $2(C_n^k)^2$; *5)* $2C_n^k \cdot C_n^{k-1}$.

Problem 4.28. *Let the graphs of the functions $y = |x-1|$, $y = |x-2|$, $y = |x-3|$, ..., $y = |x-n|$ (n graphs in total) are drawn on the coordinate plane. These graphs create the combined graph, which we denote by W. On this combined graph, the polygonal chains that intersect all vertical lines only once are considered.*

1. *How many such polygonal chains exist?*
2. *How many of them have: a) one vertex? b) two vertices? c) three vertices?*
3. *How many polygonal chains have their left and right rays coinciding with the left ray of the graph of the function $y = |x-1|$ and the right ray of the graph of the function $y = |x-n|$ respectively?*
4. *How many polygonal chains have their left and right rays coinciding with the left ray of the graph of the function $y = |x-2|$ and the right ray of the graph of the function $y = |x-(n-1)|$ respectively?*

Answer. *1)* $\frac{1}{n+2}C_{2(n+1)}^{n+1} - 1$; *2) a)* $\frac{n(n+1)}{2}$; *b) no such polygonal chains exist; c)* $\frac{1}{3}C_{n+1}^2 \cdot C_n^2$; *3)* $\frac{1}{n}C_{2(n-1)}^{n-1}$; *4)* $C_{2(n-1)}^{n-1} - C_{2(n-1)}^{n-2}$.

Hint. 1) Compare this problem with the problem about subdiagonal paths in a square.

Solution. 2c) We present one of the methods of solution on the example of the case $n = 4$ (Fig. 4.19). If a polygonal chain has 3 vertices, then 2 of them are "lower" and one is "upper". For example, x and b are lower points of the polygonal chain highlighted in Fig. 4.19), and p is its upper point. An upper point is always located between two lower points. It shares one of the right rays of the combined graph W with the left lower point, and one of the right rays of W with the right lower point. Only a point of intersection of two rays that belong to graphs of different functions $y = |x-k|$ $(k = 1, 2, ..., n)$ can be an upper point. If T is the point of intersection of rays α and β of the graph W, and M, N are some points that lay lower on these rays (one point on each ray) than T, then there exists a unique polygonal chain that contains the points T, M, and N, and is constructed of 4 line segments, two of which are rays that extend from the points M and N. The points of intersection of the rays of W, which are the only candidates for upper points, create 3 levels (in the general case, $(n-1)$ levels; one point on the highest level, then two more points, then three more, etc., finally, on the lowest level, there are $(n-1)$ points). For each point on the lowest level (in Fig. 4.19, these are the points a, b and c), we have one pair of lower points. Hence, there are 3 polygonal chains (with three vertices each) that contain the upper points. For each of

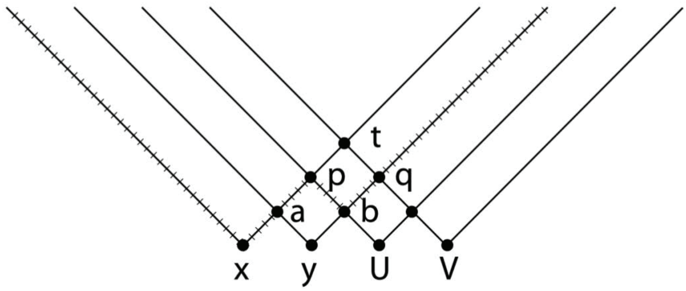

Figure 4.19. Polygonal chains with three vertices?

the upper points p and q, two lower points can be chosen in $2 \cdot 2$ ways (there are following options for p: $(a;b)$, $(a;u)$, $(x;b)$ and $(x;u)$). Therefore, there are $2 \cdot 2^2$ wanted polygonal chains with these upper points. Finally, for the upper point t residing on the third level, there are $3 \cdot 3$ ways to choose the lower points, and thus, there exist $1 \cdot 3^2$ four-segment polygonal chains containing this upper point.

Conclusion. On the combined graph in Fig. 4.19, we can find

$$\sigma_4 = 3 \cdot 1^2 + 2 \cdot 2^2 + 1 \cdot 3^2 = 20$$

polygonal chains with three vertices, each of which intersects with any vertical line once.

The above method is straightforward to apply in the case when there are n initial lines $y = |x-k|$ forming the combined graph instead of 4. In this case, the amount of wanted polygonal chains is defined by the sum

$$\sigma_n = (n-1) \cdot 1^2 + (n-2) \cdot 2^2 + (n-3) \cdot 3^2 + \ldots + 1 \cdot (n-1)^2.$$

A very essential shortcoming of this formula is that the number of summands in it increases with the growth of n. Thus, we face a serious problem: is there a way to "reduce" the formula? And if so, how can we make it?

The sum σ_n can be reduced with the help of three equalities:

$$\sum_{i=1}^{k} i = 1+2+3+\ldots+k = \tfrac{k(k+1)}{2};$$
$$\sum_{i=1}^{k} i^2 = 1^2+2^2+3^2+\ldots+k^2 = \tfrac{k(k+1)(2k+1)}{6};$$
$$\sum_{i=1}^{k} i^3 = 1^3+2^3+3^3+\ldots+k^3 = \left(\tfrac{k(k+1)}{2}\right)^2.$$

For instance, the above equalities can be proved by induction. In addition, the first of them is the special case of the familiar formula of the sum of consecutive terms of arithmetic progression.

Let us transform the sum σ_n as follows:

$$\begin{aligned}
\sigma_n = & 1^2+2^2+3^2+\ldots+(n-3)^2+(n-2)^2+(n-1)^2+\\
+\quad & 1^2+2^2+3^2+\ldots+(n-3)^2+(n-2)^2+\\
+\quad & 1^2+2^2+3^2+\ldots+(n-3)^2+\\
+ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\\
+\quad & 1^2+2^2+3^2+\\
+\quad & 1^2+2^2+\\
+\quad & 1^2.
\end{aligned}$$

Reducing the sum in each row, we get:

$$\begin{aligned}
\sigma_n &= \sum_{j=1}^{n-1} \tfrac{j\cdot(j+1)\cdot(2j+1)}{6} = \tfrac{1}{6}\cdot\sum_{j=1}^{n-1}(2j^3+3j^2+j) =\\
&= \tfrac{1}{3}\sum_{j=1}^{n-1} j^3 + \tfrac{1}{2}\sum_{j=1}^{n-1} j^2 + \tfrac{1}{6}\sum_{j=1}^{n-1} j =\\
&= \tfrac{1}{3}\cdot\left(\tfrac{(n-1)n}{2}\right)^2 + \tfrac{1}{2}\cdot\tfrac{(n-1)(2n-1)}{6} + \tfrac{1}{6}\cdot\tfrac{(n-1)n}{2} =\\
&= \tfrac{1}{12}\cdot[(n-1)^2n^2+(n-1)n(2n-1)+(n-1)n = \tfrac{(n-1)n^2(n+1)}{12}.
\end{aligned}$$

Thus, we obtain a brief direct formula that solves the problem. The formula evidence that the problem stated in this problem is solved. This formula is the final result of the current investigation. However, looking at it from a different angle, we can see that it can give rise to a new problem and new research. Express the general formula as follows:

$$\sigma_n = \frac{1}{3}\cdot\frac{(n-1)n}{2}\cdot\frac{n(n+1)}{2}.$$

The second and third factors on the right-hand side of equality are quite recognizable:

$$\frac{n(n-1)}{2} = C_n^2,\quad \frac{(n+1)n}{2} = C_{n+1}^2.$$

Hence,

$$\sigma_n = \frac{1}{3}C_n^2\cdot C_{n+1}^2.$$

As we can see, the sought value is elegantly expressed with binomial coefficients. Can you suggest a method to solve this problem and get to the last formula directly, without having to perform rather complex calculations, such as we have encountered above?

Problem 4.29. *This problem belongs to the family of problems about the subdiagonal paths in a square. We provide a brief reminder about the notions involved. Let $ACBD$ be a square with vertices in points $A\,(0;0)$, $C\,(0;n)$, $B\,(n;n)$ and $D\,(n;0)$, where n is a given natural number. We are interested in the shortest paths connecting A and B which evolve along integer-valued coordinate lines $x=k$ and $y=s$ $(k,s=0,1,2,\ldots,n)$. It has been established above that there are C_{2n}^n such paths (of length $2n$) in total. Those of them which do not extend beyond the bounds of the triangle ABD (they lay under the diagonal AB of the square $ACBD$) are called subdiagonal. We have already found that there are $\frac{1}{n+1}C_{2n}^n$*

(the Catalan number) subdiagonal paths in this square. Thus, they form $\frac{1}{n+1}$-th part of all shortest paths from A to B.

In the above setting, let k be a natural number, $1 < k < n$. How many subdiagonal paths from A to B are there, which:

1. *include the point $M(k;k)$?*

2. *include the point $N(k;k-1)$?*

Calculate the number of the above paths for $n = 6$, $k = 3$.

<u>Answer.</u> *1)* $\frac{1}{(k+1)(n-k+1)} \cdot C_{2k}^{k} \cdot C_{2(n-k)}^{n-k}$;

2) $(C_{2(k-1)}^{k-1} - C_{2(k-1)}^{k-3}) \cdot (C_{2(n-k)}^{n-k} - C_{2(n-k)}^{n-k-2})$.

Problem 4.30. *Consider the lines $y = x + k$ and $y = -x + k$ $(k = 0, \pm 1, \pm 2, \pm 3, ..., \pm s)$ on the coordinate plane, and their combined graph L. As we have determined above, there are $C_{4(s+1)}^{2(s+1)} - 2$ continuous functions defined on R, the graphs of which are parts of L (these are polygonal chains and straight lines which cross any vertical line in one point).*

Find how many of these functions are odd and how many are even.

<u>Answer.</u> *There are $2^{2s+2} - 2$ even functions, and $C_{2(s+1)}^{s+1}$ odd.*

Problem 4.31. *A road network on the coordinate plane is defined by the lines $x = k$ and $y = s$, where k and s are integers. How many ways to reach the point $B(n;n)$ departing from the point $A(0;0)$ are there, if the point $P(u,v)$ $(0 < u < n,\ 0 < v < n)$ is to be bypassed?*

<u>Answer.</u> $C_{2n}^{n} - C_{u=v}^{u} \cdot C_{2n-u-v}^{n-u}$.

Problem 4.32. *The lines $y = x + k$ and $y = -x + k$ $\quad (k = 0, \pm 1, \pm 2, \pm 3, ..., \pm s)$ form the combined graph L. How many polygonal chains, which are parts of L, are the graphs of everywhere defined functions and include the point of intersection of the lines $y = x + s$ and $y = -x - s$, as well as the point of intersection of the lines $y = -x + s$ and $y = x - s$?*

<u>Answer.</u> $4 \cdot C_{4s}^{2s}$.

Problem 4.33. *A road network in the space is created by the lines $\begin{cases} x = k, \\ y = s; \end{cases}$ $\begin{cases} x = k, \\ z = s; \end{cases}$ and $\begin{cases} y = k, \\ z = s; \end{cases}$ where k and s are integers. As we can see, every road is parallel to one of the coordinate axes and passes through an integer point of a plane that is orthogonal to it. For example, the line $\begin{cases} x = k, \\ y = s \end{cases}$ is parallel to the axis of applicants and passes through the point $(k;s)$ of the coordinate plane $z = 0$. Denote the above road network by Σ.*

1. *What is the length of the shortest path from the point $A(0;0;0)$ to the point $M(p;q;r)$ $(p, q, r$ are integer non-negative numbers)?*

2. *How many shortest paths from A to M are there?*

3. *How many shortest paths from $A(0;0;0)$ to $B(n;n;n)$ (n is natural) are there?*

4. *How many shortest paths from A to B pass through the point $P(k;k;k)$, where $0 < k < n$ and k is natural?*

5. *How many shortest paths between the points $K(a;b;c)$ and $T(u;v;t)$ (both points have integer coordinates)?*

Answer. *1)* $p+q+r$; *2)* $C^{p}_{p+q+r}\cdot C^{q}_{q+r}$; *3)* $C^{n}_{3n}\cdot C^{n}_{2n}$; *4)* $C^{k}_{3k}\cdot C^{k}_{2k}\cdot C^{n-k}_{3(n-k)}$; *5)* $C^{|a-u|}_{|a-u|+|b-v|+|c-t|}\cdot C^{|b-v|}_{|b-v|+|c-t|}$.

Problem 4.34. *Define all integer points with non-negative coordinates which belong to the plane $x+y+z=3$. For each of the above points, find the number of shortest paths that lead from it to the point of origin along the roads from Σ (see the previous problem).*

Without reference to the result of the first part of this problem, find the number of the shortest paths connecting the point of origin with the plane $x+y+z=3$.

What equality involving binomial coefficients comes as a result of the answers to the previous questions?

Answer. $C^1_3C^1_2+3(C^0_3+C^1_3+C^2_3)=3^3$.

Problem 4.35. *What is the length of the shortest path (from Σ) between the point $A(0;0;0)$ and the plane $x+y+z=n$ (n is given natural number)? How many such paths exist? How many points of the plane $x+y+z=n$ are equidistant (when moving along paths from Σ) from the point $A(0;0;0)$ and from the plane $x+y+z=n$?*

Answer. n; 3^n; C^2_{n+2}.

Hint. In order to get from the point $A(0;0;0)$ to the plane $x+y+z=n$ following the shortest possible path, one needs to increase one of the coordinates by 1 step by step. n steps are required in total. In each step there is a choice of three options.

The number of wanted points is the number of integer non-negative solutions to the equation $x+y+z=n$.

Problem 4.36. *In the setting of the road network introduced in the previous problems, find the number of shortest paths between the points $A(0;0;0)$ and $B(3;2;4)$, which partially belong to the plane $x=4$?*

Answer. $C^5_{11}\cdot C^2_6$.

Hint. Let C be the reflection of the point A w.r.t. the plane $x=4$. Establish a bijection between the wanted paths and all shortest paths between C and B.

Problem 4.37. *How many shortest paths along integer-valued coordinate lines $x=k$ and $y=s$ (k and s are integers) lead from the point $A(0;0)$ to the bounds of the square $|x|+|y|=n$ (n is given natural number)?*

Answer. $4\cdot(2^n-1)$.

Problem 4.38. *How many shortest paths along the roads from Σ (see problem 28) lead from the point $A(0;0;0)$ to the plane $|x|+|y|+|z|=n$ (n is natural)?*

Answer. $8\cdot 3-12\cdot 2^n+6$.

Problem 4.39. *In section 9 of this chapter, we have solved the problem about the game of two players that lasts until n wins of one of them. Recall the exact formulation of that problem.*

Two players participate in a competition which consists of separate matches (such as pool, tennis, etc.). By the rules of the competition, no match can end with a draw. In order to determine the winner (the champion), the players agreed to play match after match until one of them gets n wins (n is a predetermined number). Our task was to find out how many different courses of competition are possible. We have dealt with this problem and have found that there are C_{2n}^{n} different options.

The following questions develop the above topic.

1. *Let k be an integer from $[0, n-1]$. How many courses of competition are possible, where at the moment of the ultimate victory of one of the players, the other has k wins? In other words, how many ways are there for the competition to end with the score $n:k$?*

2. *Knowing the total amount of possible courses of competition, apply the results of the previous task to construct equality involving binomial coefficients.*

3. *How many courses of competition are possible, where the second player never leads by the total score?*

4. *How many courses of competition are possible, where the first player leads by the total score after each match?*

5. *How many courses of competition are possible, where the difference between the numbers of wins and losses of any player never exceeds 1?*

Answer. *1)* $2C_{n+k-1}^{k}$*; 2)* $C_{n-1}^{0} + C_{n}^{1} + C_{n+1}^{2} + \ldots + C_{2n-2}^{n-1} = \frac{1}{2}C_{2n}^{n}$*; 3)* $\frac{1}{n+1}C_{2n}^{n}$*; 4)* $\frac{1}{n}C_{2n-2}^{n-1}$*; 5)* 2^n.

Hint. 1) We can count those courses of competition where the first player (player A) becomes the champion, and the second player (player B) has k wins at the moment of the last match, and then double up the result. Proceed like that. We know that there is a "natural" bijection between the courses of competition and the shortest paths from $A(0;0)$ to $B(n;n)$. We talked about it in section 9 of this chapter. The basis for this bijection is the following observation. If we call players A and B, then any course of competition can be encoded by a sequence of these two letters. The letters A or B in one of the positions in a sequence mean that player A or player B respectively wins the corresponding match. If the competition lasts until n wins of one of the players, then the code should contain either n letters A, or n letters B, and its length should not exceed $2n-1$ (the losing player can not win more than $n-1$ matches). In addition, the last letter of the code is the one that repeats n times in it, as the competition ends after the n-th win of one of the players. Now, if we interpret the code of a course of competition as a code of a path on the coordinate plane that begins in the point $A(0;0)$ (A denotes step to the right, B denotes step up), then such path ends either on the interval BD (when player A is victorious), or on the interval CB (when player B becomes the champion). The points C and D mentioned above have coordinates $C(0;n)$, $D(n;0)$. The endpoint of a path can be any integer point of these intervals except for the point $B(n;n)$. Thus, these paths between the point $A(0;0)$ and integer points of the intervals (excluding B) can be considered to be explicit geometric illustrations of different

courses of the competition. In other words, there is a bijection between the courses and the paths. It remains to realize that every such path can be extended to the path from $A\,(0;0)$ to $B\,(n;n)$ in a unique way (adding the line segment laying on the interval DB or CB). We conclude that the paths from $A\,(0;0)$ to $B\,(n;n)$ "naturally" express different courses of the competition as well. Hence, the amounts of both are equal.

Now, ask ourselves: which paths from A to B correspond to those courses of the competition when player A becomes the champion and player B gets k points (wins k matches)? The answer is obvious: these are the paths that enter the side BD of the square $ACBD$ in the point $H\,(n;k)$. The amount of such paths is equal to the number of ways to get from the point $A\,(0;0)$ to the point $I\,(n-1;k)$ along the shortest trajectory. It remains to recall that there are C^k_{n+k-1} shortest paths leading from the point $A\,(0;0)$ to the point $I\,(n-1;k)$. Therefore, there are C^k_{n+k-1} ways for the competition to end with the score $n:k$ in favor of player A. Exactly the same amount of competitions can end with the score $n:k$ in favor of player B.

Problem 4.40. *In the previous section of this chapter and in the previous problem, we have established a relationship between competitions running until n wins of one of two participants and the shortest paths on the coordinate plane between the points $A\,(0;0)$ and $B\,(n;n)$, evolving along integer-valued coordinate paths. Here, we suggest solving two major problems concerning the competitions until n wins, without resorting to the models associated with the shortest paths.*

1. *Two players participate in a competition that consists of separate matches and runs until one of the players gets n wins. By the rules of the competition, no match can end with a draw. How many ways are there for player A to win the competition with the score $n:k \quad (k=0,1,2,...,n-1)$?*

2. *Having found the answer to the previous question, we can express the total amount possible courses of the competition with the sum*

$$2(\gamma_0+\gamma_1+\gamma_2+...+\gamma_{n-1}).$$

Here, γ_k is the answer to the previous question. Reduce this sum, applying properties of binomial coefficients.

Solution. 1. The course of competition that ends with the victory of player A and the score $n:k$ is defined by a sequence of n letters A and k letters B. In addition, the last letter of this sequence should necessarily be A. One such sequence differs from the others with the places occupied by the letter B. It can appear in any position from the first up to the $n+k-1$-th. There should be k such positions in total. Hence, there are C^k_{n+k-1} possible courses of the competition.

2. The number of possible courses of the competition is

$$2(C^0_{n-1}+C^1_n+C^2_{n+1}+...+C^k_{n+k-1}+C^{n-1}_{2n-2}).$$

Taking into account that $C^0_{n-1}=C^0_n$ and applying the equality

$$C^{s-1}_m+C^s_m=C^s_{m+1},$$

we can reduce the sum in the parentheses step by step. Finally, we arrive at

$$2C_{2n-1}^{n}.$$

Considering $C_{2n-1}^{n} = C_{2n-1}^{n-1}$, we get:

$$2C_{2n-1}^{n} = C_{2n-1}^{n-1} + C_{2n-1}^{n} = C_{2n}^{n}.$$

Problem 4.41. *Again, consider the shortest paths between the points $A\,(0;0)$ and $B\,(2s;2s)$, constructed of the line segments of the lines $x=k$ and $y=m$ (k and m are integers). How many of these paths are symmetrical with respect to:*

a) the line $y=-x+s$?

b) the point $Q\,(s;2)$?

Answer. *a) 2^s; b) C_{2s}^{s}.*

Chapter 5

Inclusion-Exclusion Principle

1. In the chapter about sets, we considered several special cases of this principle. Below, we present the general case and some typical applications of the inclusion-exclusion principle.

Knowing the amounts of elements $|A|$ and $|B|$ of (finite) sets A and B is not enough to define the number of elements of the union $A \cup B$ of these sets. The reason is on the surface: the sets A and B can share elements. Depending on the amount of common elements, the number $|A \cup B|$ may vary from $\max\{|A|, |B|\}$ (greater of the numbers $|A|$ and $|B|$) and $|A|+|B|$. For instance, if $A \subset B$, then $|A \cup B| = |B|$, as $A \cup B = B$. Alternatively, if $A \cap B = \emptyset$ (A and B do not have common elements), then $|A \cup B| = |A| + |B|$. The more elements are there in the intersection $A \cap B$, the less elements compared to $|A| + |B|$ are there in the union $A \cup B$. In fact, the connection between the numbers $|A|$, $|B|$, $|A \cup B|$ and $|A \cap B|$ is expressed by the equality

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This formula is called the inclusion-exclusion principle for two (finite) sets. It is universal in the sense that it is always correct, disregarding the power of the intersection of A and B. Fig. 5.1 illustrates the inclusion-exclusion principle schematically. The shape hatched with ascending lines denotes the set A, and the one hatched with descending lines denotes the set B. The double-hatched shape denotes the set $A \cap B$. The number $|A| + |B|$ exceeds the power of the union $A \cup B$ by $|A \cap B|$, as adding the numbers of elements of the sets A and B we account their shared elements (that is, the elements of the intersection $A \cap B$) twice. Subtracting the number $|A \cap B|$ from $|A| + |B|$, we get the number which accounts for any element of the union once. This is what the inclusion-exclusion principle states.

The formula of inclusion-exclusion principle gets more complicated with the growth of the number of sets. For three sets A, B, and C, it looks as follows

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |B \cap C| + |C \cap A|) + (|A \cap B \cap C|).$$

Now, the right-hand side contains the summands of three types. In order to stress this, we have surrounded the similar summands with parentheses. Between the first pair of parentheses, there is the sum of the number of elements of each set. The second pair of parentheses contains the sum of numbers of elements of pairwise intersections of the original sets. Finally, the number in the third pair of parentheses is the number of elements of the intersection of all three sets.

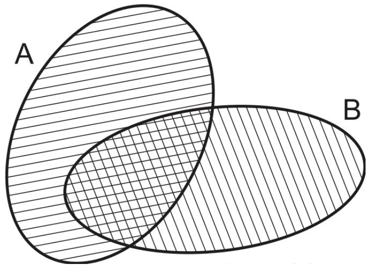

Figure 5.1. Inclusion-exclusion principle.

How to ensure that the number on the left-hand side of the equality is indeed the amount elements in the union of our sets? One possible approach is presented below. Let us assign every element of the set $A \cup B \cup C$ to one of three groups according to the number of sets to which these elements belong. Then we will ascertain that every element of any of these groups is accounted for once on the right-hand side.

Proceeding with this idea, we begin splitting the elements into groups. Let an element x belongs to one of three sets only. In this case, it is not included in any of the intersections of two or three sets. Therefore, it is accounted for once in the first pair of parentheses, and it is not accounted for in any other summands. It appears that the element x is accounted for in the right-hand side of the equality once. Now, let y belong to any two of the given sets. Then it is the element of one of three pairwise intersections, but it is not included in the intersection $A \cap B \cap C$. Therefore, it is accounted for twice between the first pair of parentheses, once between the second pair, and it is not included in the amount inside the third pair of parentheses. It appears that the element y is accounted for $2-1+0=1$ times on the right-hand side of the equality.

Finally, consider an element z which belongs to all three given sets A, B and C. Then it belongs to every pairwise intersection $A \cap B$, $A \cap C$ and $B \cap C$, and to the intersection $A \cap B \cap C$. Therefore, it is accounted for $3-3+1=1$ times on the right-hand side of the equality.

Conclusion: the formula has passed the test successfully.

The proof of the inclusion-exclusion formula for the case of three sets gives guidance on how to prove it in the general case when there are n sets. However, first, we have to properly generalize the formula developed for two and three sets.

Let $A_1, A_2, A_3, \dots, A_n$ be given sets. Call them first level blocks, and the numbers of their elements first level block numbers. Thus, the first level block numbers are $|A_1|, |A_2|, \dots, |A_n|$. There are n of them in total. Denote the sum of these numbers by σ_1.

Let us call the pairwise intersections $A_i \cap A_j$ (the intersections of two first level blocks) second level blocks. Similarly, the second level block numbers are $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, $|A_2 \cap A_3|$ etc. There are C_n^2 such numbers altogether. Clearly, some numbers can repeat. Denote their sum by σ_2.

Proceeding similarly, we call the intersections of three sets $A_i \cap A_j \cap A_k$ (there are C_n^3 of them) third level blocks and their powers third level block numbers. Let the sum of these

numbers be denoted by σ_3.

Continuing in this vein, we finally get to the last block $A_1 \cap A_2 \cap \ldots \cap A_n$ (there is only one block of this type). The number of its elements is called an n-th block number and is denoted by σ_n.

By analogy with the previous cases ($n = 2$ and $n = 3$), the general (for arbitrary n) inclusion-exclusion formula may look as follows:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \ldots + (-1)^{n-1}\sigma_n. \tag{5.1}$$

It remains to prove that the above expression is correct. To this end, choose an arbitrary element a of the union $A_1 \cup A_2 \cup \ldots \cup A_n$. We have to ensure that it is accounted for on the right-hand side of the above equality. In addition, it should be accounted for exactly one time. As a belongs to the union of our sets, it belongs to at least one of them (although it can be an element of several sets at once). Let a be an element of k of the given sets (and does not belong to another set). The symbol k may be any natural number from 1 to n. Then the element a belongs to:

exactly $C_k^1 k$ first level blocks;
exactly C_k^2 second level blocks;
exactly C_k^3 third level blocks;
and so on; finally,
exactly C_k^k k-th level blocks.
It does not belong to the blocks of higher levels.
Therefore, the element a is accounted for:
by the sum $\sigma_1 - C_k^1$ times;
by the sum $\sigma_2 - C_k^2$ times;
by the sum $\sigma_3 - C_k^3$ times;
and so on; finally,
by the sum $\sigma_k - C_k^k$ times.
Any of the rest of the sums σ_{k+1}, ..., σ_n does not account for this element.

The conclusion is that the right-hand side of the hypothetical equality accounts for the element a

$$C_k^1 - C_k^2 + C_k^3 - C_k^4 + \ldots + (-1)^{k-1}C_k^k$$

times. Now, we need to determine what this number is. Recall one of the most well-known equalities for binomial coefficients:

$$C_k^0 - C_k^1 + C_k^2 - C_k^3 + C_k^4 - \ldots + (-1)^k C_k^k = 0.$$

This equality can be derived from the binomial formula

$$(1+x)^k = C_k^0 + C_k^1 x + C_k^2 x^2 + C_k^3 x^3 + \ldots + C_k^k x^k$$

if we put $x = -1$ in it.

We can transform this equality as follows

$$C_k^0 = C_k^1 - C_k^2 + C_k^3 - C_k^4 + \ldots + (-1)^{k-1}C_k^k.$$

The last equality evidences that the expression of interest equals to 1, as $C_k^0 = 1$.

It appears that arbitrarily chosen element a from the union $A_1 \cup A_2 \cup \ldots \cup A_n$ is accounted for once in the right-hand part of the hypothetical inclusion-exclusion formula. Needless to say, the right-hand side has no relation to the elements outside the union $A_1 \cup A_2 \cup \ldots \cup A_n$. This gives grounds to claim that equality (5.1) is correct.

2. In this section, we illustrate some applications of the inclusion-exclusion principle with theoretically important examples rather than with straightforward computational problems. Thus, we will derive some less ordinary results.

First, recall several arithmetical definitions. Two natural numbers a and b are called mutually prime if their only common divisor is 1.

Example 5.1. *The numbers* 33 *and* 132 *are not mutually prime because both are divisible by* 11.

Example 5.2. *On the contrary, the numbers 21 and 55 are mutually prime. There is no common divisor greater than for these numbers.*

Example 5.3. *Can a number a be mutually prime with itself? Yes, but only if* $a = 1$.

Let n be a natural number. Let $\varphi(n)$ *denote the number of those natural numbers m which possess both following properties:*

a) $m \leq n$;

b) m and n are mutually prime.

Obviously, the symbol $\varphi(n)$ denotes a function that is defined on the set of natural numbers. Its values are also natural numbers. This function is called Euler's function by the prominent Swiss mathematician of XVIII century who studied its properties. In particular, he introduced the notation $\varphi(n)$ which eventually became conventional. The table of values of the Euler function for several initial values of the argument n is presented in the table 5.1 below.

Table 5.1. Euler's function

n	1	2	3	4	5	6	7	8	9	10	11	12	...
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	...

It takes a glance at the bottom row of the table to arrive at a disappointing conclusion: the law of correspondence between the first and the second row is not straightforward. Upon more thorough consideration, it appears that the fluctuation of values of Euler's function which seems chaotic at first sight is inevitable and absolutely natural because these values depend on the number of divisors of the argument rather than on its magnitude. The greater the amount of divisors of n is, the less there are numbers in the interval $[1, n]$ which are mutually prime with n.

It is quite simple to realize how to compute the values of the Euler function for the prime values of n. If n is prime, then (by definition) it is divisible by 1 and n, and all numbers less than it are mutually prime with it. Thus, there is the following simple formula for prime n: $\varphi(n) = n - 1$.

Talking about the general formula, there are two essentially different ways to prove it. We choose the one that uses the inclusion-exclusion principle.

We remember that any natural number except for 1 and prime numbers is the product of prime factors. For example, $6 = 2\cdot 3$, $50 = 2\cdot 55$, $38 = 2\cdot 19$, $105 = 3\cdot 5\cdot 7$, etc. Prime numbers, say, 2, 3, 5, 7, 11 and infinitely many others serve as indivisible multiplicative blocks from which any other natural number can be created (constructed) with the operation of multiplication. In addition, different combinations of prime numbers result in different numbers being constructed. This property is called the uniqueness of the prime factors decomposition (prime factorization) of natural numbers. Its meaning is the following. If two competent persons decompose the same number into prime factors independently from each other, then the obtained products will contain the same prime numbers each of which repeat in each product the same number of times. The two products can differ from each other in the order of factors only. For example, the number 12 can be expressed as $2\cdot 2\cdot 3$, or $2\cdot 3\cdot 2$, or $3\cdot 2\cdot 2$. However, any such representation will contain two 2 and one 3.

If one decomposes the number n into prime factors, places them in ascending order and replaces the products of the same factors with the corresponding powers, then the resulting numeric expression is called the canonical prime factors decomposition (canonical prime factorization) of the number n. Here are several examples of the canonical prime factorization: $6 = 2\cdot 3$, $24 = 2^3\cdot 3$, $162 = 2\cdot 3^4$, $125 = 5^3$. The notion of canonical prime factorization of a number extends on prime numbers as well, and it is assumed that any prime number has only one prime factor.

There is a reason for us to recall the well-known facts concerned with prime factors decomposition. Our strategical task is to deduce the computational formula for the Euler function. The values of this function $\varphi(n)$ of the natural argument n are inevitably related to the prime factorization of n. Therefore, it is necessary to begin with the assumption that all prime factors of n are known. There is no chance to find the formula for $\varphi(n)$ without this assumption.

Let

$$n = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$$

be the canonical prime factorization of the number n. In particular, this means that $p_1, p_2, \ldots, p_s$ are different prime numbers and $k_1, k_2, \ldots, k_s$ are natural numbers. Introduce the following notation:

A_1 is the set of those natural numbers from the interval $[1, n]$ which are divisible by p_1;
A_2 is the set of those natural numbers from the interval $[1, n]$ which are divisible by p_2;
A_3 is the set of those natural numbers from the interval $[1, n]$ which are divisible by p_3;
and so on, up to
A_s is the set of those natural numbers from the interval $[1, n]$ which are divisible by p_s.

A number t from the interval $[1, n]$ is not mutually prime with n if and only if it is divisible at least by one of the prime numbers $p_1, p_2, \ldots, p_s$. In other words, t is not mutually prime with n only when $t \in A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_s$. This is how the problem about the Euler function relates to the inclusion-exclusion formula! If we find the amount of elements of the set $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_s$, then we will be straightforward for us to determine the number $\varphi(n)$, because

$$\varphi(n) + |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_s| = n.$$

According to the inclusion-exclusion formula, we have:

$$|A_1 \cup A_2 \cup \ldots \cup A_2| = \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \ldots + (-1)^{s-1}\sigma_s.$$

If we could find the values of $\sigma_1, \sigma_2, \ldots, \sigma_s$ for our sets $A_1, A_2, \ldots, A_s$, then we would know the power of the set $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_s$, and the value of $\varphi(n)$ would follow by a simple subtraction.

The number σ_1 is the sum of all first level block numbers, which are the numbers$|A_1|$, $|A_2|$, ..., $|A_s|$. As

$$|A_i| = \frac{n}{p_i}$$

(every p_i-th number is divisible by p_i), then

$$\sigma_1 = \frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} + \ldots + \frac{n}{p_s}.$$

The number σ_2 is the sum of all first level block numbers, which are the numbers $|A_1 \cap A_2|$, $|A_1 \cap A_3|, \ldots,$ $|A_i \cap A_s|$. Due to the fact that

$$\left|A_i \cap A_j\right| = \frac{n}{p_i p_j},$$

we have

$$\sigma_2 = \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \ldots + \frac{n}{p_i p_j} + \ldots + \frac{n}{p_{s-1} p_s}.$$

Similarly,

$$\sigma_3 = \frac{n}{p_1 p_2 p_3} + \ldots$$

(the sum includes all possible products $p_i p_j p_e$ of different prime numbers and contains C_s^3 summands in total);

$$\sigma_4 = \frac{n}{p_1 p_2 p_3 p_4} + \ldots$$

(the sum includes all possible products $p_i p_j p_e p_r$ of different prime numbers and contains C_s^4 summands in total).

Repeating the above procedure, we can construct all other sums of block numbers. The last of them consists of only one summand:

$$\sigma_s = \frac{n}{p_1 p_2 \cdots p_s}.$$

We have derived the formula for $\varphi(n)$:

$$\varphi(n) = n - \sigma_1 + \sigma_2 - \sigma_3 + \ldots + (-1)^s \sigma_s =$$

$$= n \cdot \left[1 - \left(\frac{1}{p_1} + \frac{1}{p_2} + \ldots\right) + \left(\frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \ldots\right) - \right.$$
$$\left. - \left(\frac{1}{p_1 p_2 p_3} + \frac{1}{p_1 p_2 p_3} + \ldots\right) + \ldots + (-1)^s \left(\frac{1}{p_1 p_2 \cdots p_s}\right)\right].$$

It gains much more attractive form if we spot the product

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_s}\right)$$

inside the square brackets.

Finally, we get:

$$\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_s}\right),$$

where $p_1, p_2, \ldots, p_s$ are different prime divisors of n.

Example 5.4. *The prime factorization of the number* 252 *is* $252 = 2^2 \cdot 3 \cdot 31$. *Therefore,* $\varphi(252) = 252 \cdot \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{31}\right) = \frac{252 \cdot 1 \cdot 2 \cdot 30}{2 \cdot 3 \cdot 31} = 120$. *There are* 120 *natural numbers in the interval* $[1, 252]$ *that have common divisors greater than* 1 *with the number* 252.

Example 5.5. *The number* 256 *is the power of* 2, *hence,* $\varphi(256) \cdot \left(1 - \frac{1}{2}\right) = 128$.

Example 5.6. *The number* 97 *is prime, hence,* $\varphi(97) = 97 \cdot \left(1 - \frac{1}{97}\right) = 96$.

3. Below, there is another example of efficient application of the inclusion-exclusion formula related to permutations.

Consider the permutations of n initial natural numbers $1, 2, 3, \ldots, n-1, n$. We will establish that there are $n!$ such permutations in total. One of them is the permutation $(1, 2, 3, 4, \ldots, n-1, n)$. This is the only permutation that has every number standing in its "natural" position: number 1 is in the first position, number 2 is in the second, and so on. In other permutations, at least one number (in fact, at least two numbers) is not in its "authentic" position, that is, not in the position which has the same number. Are there any permutations where no numbers stand in their "natural" positions? Let us consider permutations of lengths 2 and three. Among the permutations of numbers 1 and 2, there is one such permutation, namely $(2, 1)$, and there are two permutations of numbers $1, 2$ and 3 of this type: $(2, 3, 1)$ and $(3, 1, 2)$.

So, how many such permutations are there among $n!$ permutations of n initial natural numbers? This is the question which we are going to answer. Begin with the proper and complete formulation of the above question.

Consider the permutations of the numbers $1, 2, \ldots, n$. Question: how many of them have all these numbers not standing in their natural positions, i.e., 1 is not in the first position, 2 is not in the second, 3 is not in the third, and so on up to n which is also not in n-th position?

Denote the number in question by $j(n)$. We attempt to find its value as a difference between the total number of permutations $(n!)$ and the number of those permutations in which at least one number stands in its natural position. The latter quantity (the number of permutations that do not shuffle all numbers) can be determined with the help of the inclusion-exclusion principle.

To this end, we order the following sets:

A_1 the set of those permutations of n initial numbers which leave 1 in its place;

A_2 the set of those permutations of n initial numbers which leave 2 in its place;

and so on up to

A_n the set of those permutations of n initial numbers which leave n in its place.

The union of these sets $A_1 \cup A_2 \cup \ldots \cup A_n$ is composed of all those permutations that do not change the place of at least one number. Therefore,

$$j(n) = n! - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

Provision of information about the number of elements of the union of sets is what the inclusion-exclusion formula is supposed to do. Appeal to this formula.

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sigma_1 - \sigma_2 + \sigma_3 - \ldots + (-1)^{n-1}\sigma_n.$$

Now, we need to calculate the sums σ_1, σ_2, ..., σ_n.

1. Every set A_i (a first level block set) contains $(n-1)!$ elements (permutations). Really, A_i is the set of those permutations that do not affect the position of the element (number) i. They can reposition other numbers without any restrictions. Therefore, there are $(n-1)!$ such permutations. We can conclude that

$$\sigma_1 = |A_1| + |A_2| + \ldots + |A_n| = n \cdot (n-1)! = n!$$

2. All second-level block numbers are also equal. Indeed, the set $A_i \cap A_k$ is composed of all those permutations that have the numbers i and k in their natural positions. There are $(n-2)!$ of them. And there are C_n^2 such sets. Hence,

$$\sigma_2 = C_n^2 \cdot (n-2)! = \frac{n!}{2!}.$$

3. In a similar fashion we ascertain that

$$\begin{array}{l} \sigma_3 = C_n^3 \cdot (n-3)! = \frac{n!}{3!}, \\ \sigma_4 = C_n^4 \cdot (n-4)! = \frac{n!}{4!}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sigma_n = C_n^n \cdot (n-n)! = \frac{n!}{n!}. \end{array}$$

Combining all the above equalities, we get:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = n! - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \ldots + (-1)^{n-1} \cdot \frac{n!}{n!}.$$

The wanted number $j(n)$ is expressed by the formula:

$$j(n) = n! \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

In particular, it can be applied to construct the table of values of the function $j(n)$ 5.2 for small values of n:

Table 5.2. Values of function $j(n)$.

n	2	3	4	5	6	7	8	9	10
$j(n)$	1	2	9	44	265	1844	14753	132776	1327761

Problems

Problem 5.1. *Suppose that the sets A, B, C and D are such that:* $|A| = 12$, $|B| = 10$, $|C| = 9$, $|D| = 8$ $|A \cap B| = |A \cap C| = |A \cap D| = |B \cap C| = |B \cap D| = |C \cap D| = 2$, $|A \cap B \cap C| = |A \cap B \cap D| = |A \cap C \cap D| = 1$, $|B \cap C \cap D| = 2$, $|A \cap B \cap C \cap D| = 1$. *Find the number of elements of the following sets:* $A \cup B$, $A \cup C$, $A \cup D$, $B \cup C$, $B \cup D$, $C \cup D$, $A \cup B \cup C$, $A \cup B \cup D$, $B \cup C \cup D$, $A \cup B \cup C \cup D$.

Answer. $|A \cup B| = 20$, $|A \cup C| = 19$, $|A \cup D| = 18$, $|B \cup C| = 17$, $|B \cup D| = 16$, $|C \cup D| = 15$, $|A \cup B \cup C| = 26$, $|A \cup B \cup D| = 25$, $|B \cup C \cup D| = 23$, $|A \cup B \cup C \cup D| = 31$.

Problem 5.2. *Prove that for any finite sets the following equality holds:*

$$|A \cup B| = |A \backslash B| + |B \backslash A| + |A \cap B|.$$

Problem 5.3. *Ascertain that for any finite sets the following equality holds*

$$|A \cap B| = (|A| + |B|) - |A \cup B|.$$

Problem 5.4. *The equality from the previous problem is an arithmetic corollary of the inclusion-exclusion formula for two sets. In order to get it, one needs to replace the summands $|A \cup B|$ and $|A \cap B|$ in the inclusion-exclusion formula*

$$|A \cup B| = (|A| + |B|) - |A \cap B|.$$

However, the equality

$$|A \cap B| = (|A| + |B|) - |A \cup B|$$

can be derived formally if we replace the "$\cup$" sign with the "$\cap$" sign and vice versa in the inclusion-exclusion formula. Performing the same manipulation (replacing the signs "$\cup$" and "$\cap$") in the inclusion-exclusion formula for three sets, we get

$$|A \cap B \cap C| = (|A| + |B| + |C|) - (|A \cup B| + |A \cup C| + |B \cup C|) + |A \cup B \cup C|.$$

Prove that this equality is also correct (note that the last equality can not be obtained from the inclusion-exclusion formula by arithmetic operations).

Hint. Split all elements into four classes: elements that do not belong to any of the sets A, B and C; elements which belong to one of these sets; elements which belong to two of these sets; elements which belong to all these sets. Count how many times the expression on the right-hand side accounts for elements of each of the above classes.

Problem 5.5. *Express the inclusion-exclusion formula as follows:*

$$\begin{aligned}
&|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = (|A_1| + |A_2| + |A_3| + \ldots + |A_n|) - \\
&-(|A_1 \cap A_2| + \ldots) + (|A_1 \cap A_2 \cap A_3| + \ldots) - \\
&-(|A_1 \cap A_2 \cap A_3 \cap A_4| + \ldots) + \ldots + \\
&+(-1)^{k-1}(|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{k-1} \cap A_k| + \ldots) + \ldots + \\
&+(-1)^{n-1}(|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|).
\end{aligned}$$

Here:

1) $A_1, A_2, A_3, \ldots, A_n$ *are arbitrary finite sets;*

2) the first pair of parentheses on the right-hand side contains the sum of all first level block numbers (recall that this means the powers of the sets A_i $(i = 1, 2, .., n)$*;*

3) the second pair of parentheses in the right-hand side contains the sum of all second-level block numbers (recall that this way call the powers of the sets $A_i \cap A_j$*;* $i \neq j$*,* $i = 1, 2, \ldots, n$*;* $j = 1, 2, \ldots, n$*);*

4) the third pair of parentheses on the right-hand side contains the sum of all third-level block numbers (which is the powers of the intersections $A_p \cap A_q \cap A_t$ *(*$A_1 \cap A_2 \cap A_3$ *is just one of them, and there are* C_n^3 *of them in total); and so on, up to the last pair of parentheses, which contains the n-level block number (there is only one such number; it expresses the number of elements of the intersection of all sets* A_k $(k = 1, 2, \ldots, n)$*).*

Replacing the signs "$\cup$*" with the signs "*$\cap$*" and vice versa, we get the equality:*

$$
\begin{aligned}
&|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n| = (|A_1| + |A_2| + |A_3| + \ldots + |A_n|) - \\
&-(|A_1 \cup A_2| + \ldots) + (|A_1 \cup A_2 \cup A_3| + \ldots) - \\
&-(|A_1 \cup A_2 \cup A_3 \cup A_4| + \ldots) + \ldots + \\
&+(-1)^{k-1}(|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{k-1} \cup A_k| + \ldots) + \ldots + \\
&+(-1)^{n-1}(|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n|).
\end{aligned}
$$

It appears that this equality is also correct. Prove this fact.

The inclusion-exclusion formula evidence that: if the numbers of elements of several sets and all their intersections are known, then the number of elements of the union of these sets is also known.

The new (dual) formula reads: if the numbers of elements of several sets and all their unions are known, then the number of elements of the intersection of these sets is also known.

In brief: the powers of intersections of finite sets define the powers of their unions and vice versa.

Proof. According to the definition of the intersection of sets, the left part of (hypothetical) equality accounts only for the elements that are included in each of the given sets A_i $(i = 1, 2, 3, \ldots, n)$. Hence, the proof would be complete if we could ascertain that the right-hand side of the equality accounts every such element once and does not account for any other elements.

First, assume that a does not belong to any of the sets A_i $(i = 1, 2, 3, \ldots, n)$. Then this element is not accounted for by any of the expressions between the parentheses on the right-hand side of the equality, and thus, the whole right-hand side takes no account of this element.

Proceed with another element a which belongs to k of n given sets $A_1, A_2, \ldots, A_n$ $(1 \leq k < n)$. In other words, consider the case when the element a is an element of some k $(1 \leq k < n)$ of our sets A_i and does not belong to the remaining $n - k$ of these sets.

The expression between the first pair pf parentheses $(|A_1| + |A_2| + \ldots + |A_n|)$ accounts for this element k times. The individual summands in the expression surrounded by the second pair of parentheses account for the element a either 1, or 0 times. The summand

$|A_p \cup A_q|$ does not account for the element a if and only if $a \notin A_p$ and $a \notin A_p$. There are C_{n-k}^2 of such summands. The rest of summands (their number is $C_n^2 - C_{n-k}^2$) account for the element a one time each. Therefore, the second pair of parentheses account for this element $(C_n^2 - C_{n-k}^2)$ times. The similar formula can be used to express the corresponding result with respect to the first pair of parentheses, as $k = C_n^1 - C_{n-k}^1$.

Move on to the third pair of parentheses. The union of three sets $A_p \cup A_q \cup A_r$ does not contain the element a if and only if none of the sets A_p, A_q and A_r contain it. Overall, there are C_{n-k}^3 such unions. The rest of the unions of three sets A_i $(i = 1, 2, 3, ..., n)$ include the element a. There are $C_n^3 - C_{n-k}^3$ such unions in total. This means that the expression surrounded by the third pair of parentheses accounts for the element a $(C_n^3 - C_{n-k}^3)$ times. The above method can be applied further: the fourth pair of parentheses account for the element a $(C_n^3 - C_{n-k}^4)$ times, the fourth pair – $(C_n^3 - C_{n-k}^5)$ times, and so on. Proceeding similarly, we will eventually reach the $(n-k)$-th pair of parentheses. The expression inside them cconts for the element a $(C_n^{n-k} - C_{n-k}^{n-k})$ times. All subsequent pairs of parentheses contain summands each of which accounts for the element a once, therefore:

$(n-k+1)-$th pair of parentheses accounts for the element a C_n^{n-k+1} times;

$(n-k+2)-$th pair of parentheses accounts for the element a C_n^{n-k+1} times;

and so on; finally,

$n-$ th pair of parentheses accounts for the element a C_n^n times.

It appears that the entire right-hand side of the hypothetical equality accounts for this element

$$(C_n^1 - C_{n-k}^1) - (C_n^2 - C_{n-k}^2) + ... + (-1)^{n-k-1}(C_n^{n-k} - C_{n-k}^{n-k}) +$$

$$+(-1)^{n-k}C_n^{n-k+1} + (-1)^{n-k+1}C_n^{n-k+2} + ... + (-1)^{n-1}C_n^n$$

times. We need to find the number expressed with this sum. Transform it as follows:

$$(C_n^1 - C_{n-k}^1) - (C_n^2 - C_{n-k}^2) + (C_n^3 - C_{n-k}^3) - ... + (-1)^{n-k-1}(C_n^{n-k} - C_{n-k}^{n-k}) +$$

$$+(-1)^{n-k}C_n^{n-k+1} + (-1)^{n-k+1}C_n^{n-k+2} + ... + (-1)^{n-1}C_n^n =$$

$$= [C_n^1 - C_n^2 + C_n^3 - ... + (-1)^{n-k-1}C_n^{n-k} + (-1)^{n-k}C_n^{n-k+1} + ... + (-1)^{n-1}C_n^n] -$$

$$-[C_{n-k}^1 - C_{n-k}^2 + C_{n-k}^3 - ... + (-1)^{n-k-1}C_{n-k}^{n-k}].$$

Both pairs of square brackets contain the similar sums, which are equal to 1, as suggested by the familiar equality

$$C_s^0 - C_s^1 + C_s^2 - C_s^3 + ... + (-1)^s C_s^s = 0$$

and the fact that $C_s^0 = 1$.

On aggregate, the right-hand side accounts for the element a 0 times, as it should be in the case our equality is correct.

It remains to consider the case when the element a is present in all given sets $A_1, A_2, ..., A_n$.

This time the element a is included in all unions of the sets A_i $(i = 1, 2, 3, ..., n)$, and thus:

is accounted for C_n^1 times by the first pair of parentheses;

C_n^2 times by the second pair;

C_n^3 times by the third pair;

and so on, finally,

C_n^n times by the last pair.

Overall, the right-hand side of hypothetical equality accounts for the element a

$$C_n^1 - C_n^2 + C_n^3 - C_n^4 + ... + (-1)^{n-1} C_n^n$$

times. As we know, this sum equals 1.

We have ensured that the right-hand side of the equality accounts only for the elements of the setsA_i $(i = 1, 2, 3, ..., n)$, as well as the left-hand side. Therefore, equality is correct. □

Problem 5.6. *How many permutations of the numbers* $1,2,3,\ldots,n$ *have exactly one number standing in its natural position?*

Answer. $n \cdot j(n-1) = n! \left(\frac{1}{2!} - \frac{1}{3!} + ... + (-1)^{n-1} \frac{1}{(n-1)!} \right)$.

Problem 5.7. *How many permutations of the numbers* $1,2,3,\ldots,n$ *are there, where exactly k numbers stand in their natural positions?*

Answer. $C_n^k \cdot j(n-k) = C_n^k \cdot (n-k)! \cdot \left(\frac{1}{2!} - \frac{1}{3!} + ... + (-1)^{n-k} \cdot \frac{1}{(n-k)!} \right)$.

Problem 5.8. *How many permutations of the numbers* $1,2,3,\ldots,n$ *are there, where none of the numbers stands in its natural position, and the first position is occupied by the number equal to the number of the position in which the number 1 stands?*

Answer. $(n-1) \cdot j(n-2)$.

Problem 5.9. *How many permutations of the numbers* $1,2,3,\ldots,n$ *are there, where none of the numbers stands in its natural position, and the first position is not occupied by the number equal to the number of the position in which the number 1 stands?*

Answer. $(n-1) \cdot j(n-1)$.

Solution. *First, count the permutations that have the number s in the initial position (while satisfying other stated conditions). As the number 1 does not stand in the s-th position (by the condition), then replacing it (number 1) with the number s and then detaching the initial number s, we get the permutation of the numbers* $2,3,4,\ldots,s,\ \ldots,\ n$, *in which none of the numbers stands in its original position:* 2 *is in the first position,* 3 *is in the second, and so on. There are* $j(n-1)$ *such permutations. Taking into account that s can gain the values of* $2,3,4,\ldots,n$, *we get the answer to the question of the problem.*

Problem 5.10. *As before, the symbol* $j(n)$ *denotes the number of those permutations of the numbers* $1,2,3,\ldots,n$ *in which none of the numbers stand in their natural positions. Basing on the results of two previous problems, prove that the following recursive formula holds for the numbers* $j(n)$*:*

$$j(n) = (n-1) \cdot (j(n-1) + j(n-2)).$$

Hint. *No permutations are satisfying the conditions of both problem 8 and problem 9. Moreover, taken together with the permutations considered in problems 8 and 9 create the set of all permutations of the numbers $1,2,3,\ldots,n$ in which none of the numbers stands in its natural positions.*

Problem 5.11. *Prove the recursive formula*

$$j(n) = (n-1)\cdot(j(n-1)+j(n-2))$$

formally-arithmetically, basing on the direct formula

$$j(n) = n!\left(\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\cdot\frac{1}{n!}\right).$$

Proof. According to the direct formula for $j(n)$, we have:

$$(n-1)\cdot(j(n-1)+j(n-2)) =$$
$$= (n-1)\cdot\Big[(n-1)!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\cdot\tfrac{1}{(n-2)!}+\right.$$
$$\left.+(-1)^{n-1}\cdot\tfrac{1}{(n-1)!}\right)+(n-2)!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\tfrac{1}{(n-2)!}\right)\Big] =$$
$$= n!\left[\tfrac{n-1}{n}\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\tfrac{1}{(n-2)!}+(-1)^{n-1}\tfrac{1}{(n-1)!}\right)+\right.$$
$$\left.+\tfrac{1}{n}\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\tfrac{1}{(n-2)!}\right)\right] =$$
$$= n!\left[\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\tfrac{1}{(n-2)!}+\tfrac{n-1}{n}\cdot(-1)^{n-1}\tfrac{1}{(n-1)!}\right] =$$
$$= n!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-2}\tfrac{1}{(n-2)!}+(-1)^{n-1}\tfrac{1}{(n-1)!}+(-1)^{n}\tfrac{1}{n!}\right) = j(n).$$

□

Problem 5.12. *Prove that $j(n)$ satisfies the recurrence relation*

$$j(n) = n\cdot j(n-1)+(-1)^n.$$

Proof. We have:

$$n\cdot j(n-1)+(-1)^n = n\cdot(n-1)!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-1}\tfrac{1}{(n-1)!}\right)+(-1)^n =$$
$$= n!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-1}\tfrac{1}{(n-1)!}\right)+n!\cdot(-1)^n\tfrac{1}{n!} =$$
$$= n!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\ldots+(-1)^{n-1}\tfrac{1}{(n-1)!}+(-1)^n\tfrac{1}{n!}\right).$$

□

Problem 5.13. *Let $i(n)$ be the number of permutations of the numbers $1, 2, 3, \ldots, n$, in which all the numbers stand in their original positions or the neighboring ones (that is: the number 1 stands in the first or second position; 2 is in the first, second or third position; 2 is in the second, third or fourth position; and so on. Define the sequence $i(n)$ with a recurrence relation and the initial conditions.*

Answer. $i(n) = 1,\ i(2) = 2;\ i(n) = i(n-1) + i(n-2)$ *for* $n \geq 3$.

Hint. Split the wanted permutations of the numbers $1, 2, 3, \ldots, n$ into two groups: permutations with the number n in the last position, and permutations with n in the penultimate position.

Problem 5.14. *How many permutations of the numbers $1, 2, 3, \ldots, n$ are there, where every number stands in the neighboring position?*

Answer. *One permutation if n – is even; none if n is odd.*

Problem 5.15. *Calculate the values of the Euler function $\varphi(n)$ for the numbers 216, 81, 93 and 625.*

Answer. 72, 54, 60, 500.

Problem 5.16. *How many (natural) numbers are mutually prime and:*

1) belong to the interval $[1, 375]$;

2) belong to the interval $[1, 750]$?

Answer. *1)* 200; *2)* 400.

Problem 5.17. *How many numbers less than 675 have the following greatest common divisor of:*

1) 5,

2) 3,

with this number?

Answer. *1)* $\varphi(135)$; *2)* $\varphi(225)$.

Problem 5.18. *How many ways are there to place 8 rooks of the same color on a chessboard so that none of them attacks each other and none of them stand on a white diagonal (connecting the left bottom corner and the top right corner)? Be aware that two rooks attack each other if and only if they stand in the same rank or file (horizontal or vertical stripe of cells).*

Answer. 14833.

Hint. Let us enumerate the files of the chessboard from left to right and the ranks from bottom to end with the numbers 1, 2, 3, 4, 5, 6, 7, 8. This introduces the coordinate system on the chessboard. Every cell (square) of the chessboard is defined by two numbers $(p; q)$. The first of them denote the corresponding file and the second denotes the rank. For example, $(3; 7)$ are the coordinates of the square located in the intersection of the third file and the seventh rank. The squares of the forbidden diagonal differ from the other squares in that they have two components the same.

Suppose the rooks have been placed on the chessboard according to the stated rules. Writing down the coordinates of the squares occupied by the rooks from left to right, we get the following sequence of pairs of numbers:

$$(1; r_1),\ (2; r_2),\ (3; r_3),\ \ldots,\ (8; r_8).$$

Here, the numbers $r_1, r_2, r_3, ..., r_8$ create a certain permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, in which the numbers do not stand in their natural positions as $r_1 \neq 1,\ r_2 \neq 2, r_3 \neq 3, ..., r_8 \neq 8$. Clearly, there is a bijection between such permutations and the placements of eight rooks on a chessboard. Therefore, there are $j(8)$ ways to place the rooks in line with the requirements.

Problem 5.19. *One needs to choose k out of given n consecutive natural numbers* 1, 2, 3, ..., *n, so that there are no adjacent numbers among them. How many ways are there to make it?*

Answer. C_{n-k+1}^{k}.

Solution. First Approach. First of all, we have to remark that adhering to the restrictions of the problem, it is possible to choose k numbers only if k is much less than n. It is straightforward to find the exact maximum ratio of k to n. Choosing any k numbers we have to exclude at least $k-1$ numbers that stand between the chosen ones. Therefore, $n \geq 2k-1$. Further, we assume that this condition is fulfilled.

Along with the sequence of n numbers

$$1, 2, 3, 4, 5, ..., n-2, n-1, n, \tag{5.2}$$

consider the sequence of $n-(k-1)$ numbers

$$1, 2, 3, 4, 5, ..., n-(k-2), n(k-1). \tag{5.3}$$

Let

$$(\alpha_1; \alpha_2; \alpha_3; ...; \alpha_k)$$

be k different elements of the sequence (5.3) placed in ascending order. Then

$$(\alpha_1; \alpha_2+1; \alpha_3+2; ...; \alpha_k+(k-1))$$

is a set of k different numbers of the sequence (5.2). In addition, these numbers are such that the difference between any two of them is greater or equal to 2. Let us ensure that this is true.

First, it is obvious that the numbers $\alpha_1, \alpha_2+1, ..., \alpha_k+(k-1)$ increase from left to right because this is the property of the numbers $\alpha_1, \alpha_2, ..., \alpha_k$. Secondly, the greatest of the numbers $\alpha_k+(k-1)$ does not exceed n as $\alpha_k \leq n-(k-1)$. Finally,

$$[\alpha_i+(i-1)]-[\alpha_{i-1}+(i-2)] = (\alpha_i-\alpha_{i-1})+1 \geq 2,$$

because $\alpha_i - \alpha_{i-1} \geq 1$.

It appears that the set (k-sequence) of numbers

$$(\alpha_1; \alpha_2+1; \alpha_3+2; ...; \alpha_k+(k-1))$$

is one of sought sets.

Now, let

$$(\beta_1; \beta_2; \beta_3; ..., \beta_k)$$

be an increasing sequence constructed of the elements of sequence (5.2), such that satisfies the condition in the statement of the problem. This means that the difference between any two adjacent elements of the sequence is not less than 2 :

$$\beta_i - \beta_{i-1} \geq 2 \ (i = 2, 3, ..., k).$$

Consider the set of numbers

$$(\beta_1; \beta_2 - 1; \beta_3 - 2; ...; \beta_k - (k-1)).$$

They possess two important features. First, these numbers increase from left to right because

$$[\beta_i - (i-1)] - [\beta_{i-1} - (i-2)] = \beta_i - \beta_{i-1} - 1 \geq 1,$$

as $\beta_i - \beta_{i-1} \geq 2$ $(i = 2, 3, ..., k)$. Second, the last (the greatest) number $\beta_k - (k-1)$ does not exceed $n - (k-1)$ because $\beta_k \leq n$. These properties evidence that

$$\{\beta_1, \beta_2 - 1, \beta_3 - 2, ..., \beta_k - (k-1)\}$$

is a k-element subset of the set (5.3).

We have established a bijection between all k-element subsets of the set (5.3) and those k-element subsets of the set (5.2) which do not contain two consecutive natural numbers. There are $n - (k-1)$ elements in the set (5.3), hence, it has

$$C^k_{n-(k-1)}$$

$k-$element subsets. According to the established bijection, this number is also the answer to the question of the problem.

Let us illustrate the above bijection, which solved the problem for us, with an exact example.

Let

$$A = \{1, 2, 3, 4, 5, 6, 7\},$$

and we are concerned with those three-element subsets of this set that do not contain two (or three) consecutive natural numbers. The examples of such subsets are $\{1, 3, 6\}$, $\{1, 4, 7\}$ or $\{2, 4, 6\}$. We have established that a bijection exists between these subsets and 3-element subsets of the set

$$B = \{1, 2, 3, 4, 5\}.$$

Below, there is a complete list of 3-element subsets of the set B and their correspondences, which are those subsets of the set A that consist of three elements and do not include two consecutive natural numbers:

Subsets of B. Subsets of A

$$\begin{array}{rcl}
\{1,2,3\} & \leftrightarrow & \{1,3,5\} \\
\{1,2,4\} & \leftrightarrow & \{1,3,6\} \\
\{1,2,5\} & \leftrightarrow & \{1,3,7\} \\
\{1,3,4\} & \leftrightarrow & \{1,4,6\} \\
\{1,3,5\} & \leftrightarrow & \{1,4,7\} \\
\{1,4,5\} & \leftrightarrow & \{1,5,7\} \\
\{2,3,4\} & \leftrightarrow & \{2,4,6\} \\
\{2,3,5\} & \leftrightarrow & \{2,4,7\} \\
\{2,4,5\} & \leftrightarrow & \{2,5,7\} \\
\{3,4,5\} & \leftrightarrow & \{3,5,7\}
\end{array}$$

Second Approach. Let us attempt solving this problem using an essentially different approach that does not exploit the idea of bijection. The sought numbers depend on two variables n and k, so it is convenient to denote them by some expression of the form $t(n, k)$. Thus, further, $t(n, k)$ denotes the number which expresses the amount of ways to choose k elements of the set

$$A = \{1, 2, 3, ..., n\},$$

so that there are no two (or more) consecutive numbers among the chosen elements.

As above, we begin with a remark that the choice with the above restrictions can only be made if $n \geq 2k-1$. Our task is to find a computational formula for $t(n, k)$ for the cases when the latter inequality holds. If $n < 2k-1$, then it is appropriate to assume that $t(n, k)$=0, because in this case it is impossible to choose k elements of the set A among which there are no consecutive numbers (in other words, these k elements should differ from each other by 2 or more).

Our wealth of experience allows supposing that valuable information about numbers of the form $t(n,k)$ can be obtained through examination of its recurrence relations. Proceed with this idea. We split all subsets of the set A into two groups. The first will contain all those subsets that include the number 1, and the second consists of all other subsets. Question ourselves: how many subsets fall in each group? Creating a subset, if we include the number 1 in it, then we will surely not include 2 in it, as any set should not contain two consecutive elements. Thus, this subset is to be supplemented with the numbers from the set $\{3, 4, ..., n\}$. We have to choose $k-1$ of them, ensuring that all of them differ from each other at least by 2. Therefore, the first group contains $t(n-2, k-1)$ subsets. The subset of the second group are all those k-element subsets of the set $\{2, 3, ..., n\}$ which do not contain two consecutive numbers. According to our notation, there are $t(n-1, k)$ such subsets. This competes the construction of the recurrence relation for the sought number $t(n,k)$:

$$t(n,k) = t(n-1,k) + t(n-2,k-1).$$

This is an undeniable success and a good sign. We can hope to find the direct computational formula for $t(n,k)$ eventually. However, there is still a long way to go. First of all, we need to find the starting point of the recurrence relation. As we remember, the recursive formula will fail to provide the expected results without initial conditions. Moreover, our formula is quite complex, it depends on two parameters (n and k) which participate in

recursion (reduction to variables with smaller values of parameters). We should investigate thoroughly the mechanics of recurrence in our case.

First, we take the smallest possible value of k, namely, $k = 1$. There is no doubt that for any natural n, we have:

$$t(n, 1) = n.$$

We consider all such equalities as initial conditions. In addition, initial conditions include the equalities

$t(n, k) = 0$ for $n < 2k - 1$.

Let us check if the recurrence relation works when $k = 2$. Are we able to use it accompanied with the initial conditions to determine the values of $t(n, 2)$ step be step for $n = 2 \cdot 2 - 1 = 3, 4, 5, \ldots$? We have:

$$\begin{aligned}
&t(3,2) = t(2,2) + t(1,1) = 0 + 1 = 1;\\
&t(4,2) = t(3,2) + t(2,1) = 1 + 2 = 3;\\
&t(5,2) = t(4,2) + t(3,1) = 3 + 3 = 6;\\
&t(6,2) = t(5,2) + t(4,1) = 6 + 4 = 10;\\
&t(7,2) = t(6,2) + t(5,1) = 10 + 5 = 15;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

No doubt, potentially, we can continue the above calculations infinitely, increasing step by step the value of n (the first parameter of recursion) by 1. Calculating the value of $t(n+1, 2)$ we make use of the value of $t\,(n, 2)$ calculated on the previous step and of the initial condition $t(n-1, 1) = n-1$.

Having done that, we will be able to calculate the values of $t(n, 3)$ step by step (in line with the growth of n). Here is the beginning of this infinite chain:

$$\begin{aligned}
&t(5,3) = t(4,3) + t(3,2) = 0 + 1 = 1;\\
&t(6,3) = t(5,3) + t(4,2) = 1 + 3 = 4;\\
&t(7,3) = t(6,3) + t(5,2) = 4 + 6 = 10;\\
&t(8,3) = t(7,3) + t(6,2) = 10 + 11 = 20;\\
&t(9,3) = t(8,3) + t(7,2) = 20 + 15 = 35;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Then, it comes turn of $t(n, 4)$. This time, the chain begins as follows:

$$\begin{aligned}
&t(7,4) = t(6,4) + t(5,3) = 0 + 1 = 1;\\
&t(8,4) = t(7,4) + t(6,3) = 1 + 4 = 5;\\
&t(9,4) = t(8,4) + t(7,3) = 5 + 10 = 15;\\
&t(10,4) = t(9,4) + t(8,3) = 15 + 20 = 35;\\
&t(11,4) = t(10,4) + t(9,3) = 35 + 35 = 70;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The results obtained during the above procedure can be expressed in the form of a table 5.3

In the intersection of the n-th column and the k-th row, there is the number $t(n, k)$. A thorough inspection of the above table inevitably suggests there is a connection with binomial coefficients. What type of connection is that? Each of the above numbers is a

Table 5.3. Values of $t(n,k)$

$k\backslash n$	1	1	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	0	0	1	3	6	10	15	...	...	...	...
3	0	0	0	0	1	4	10	20	35	...	...
4	0	0	0	0	0	0	1	5	15	35	70

binomial coefficient, although the hypothesis of $t(n,k) = C_n^k$ fails immediately. In order to figure out the "true" hypothesis about the relation between the numbers $t(n,k)$ and the binomial coefficients C_n^k, it will be helpful to construct a similar table (for the same values of n and k) of C_n^k to compare with the table of numbers $t(n,k)$. Here is this table:

Table 5.4. Values of C_n^k

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	0	1	3	6	10	15	...	...	...	...	...
3	0	0	1	4	10	20	35	...	...	...	...
4	0	0	0	1	5	15	35	70	...	....	...

The similarity of both tables 5.3 and 5.4 is unquestionable. Their first rows are identical, They evidence that

$$t(n,1) = C_n^1.$$

The second row of the second table replicates the second row of the first one offsetting its values by one step. This means that

$$t(n,2) = C_{n-1}^2.$$

Similarly, the third row of the second table is actually the third row of the first table offset by two cells to the left. Thus

$$t(n,3) = C_{n-2}^3.$$

Finally, comparing the fourth rows of both tables, we discover that

$$t(n,4) = C_{n-3}^4.$$

We get the following vocabulary for the translation of the values of $t(n,k)$ into the familiar language of C_s^r:

It is easy to spot the pattern:

$$t(n,k) = C_{n-(k-1)}^k.$$

Table 5.5. Correspondence $t(n,k)=C^{k}_{n-(k-1)}$

$t(n,k)$	$t(n,1)$	$t(n,2)$	$t(n,3)$	$t(n,4)$	$\dots$
C^{r}_{s}	C^{1}_{n}	C^{2}_{n-1}	C^{3}_{n-2}	C^{4}_{n-3}	$\dots$

It remains to prove this hypothetical direct formula for $t(n,k)$. We have a wealth of experience in this domain and appreciate what exactly we are about to prove. First of all, we have to ascertain that the initial conditions for $t(n,k)$ are fulfilled for our formula. Recall them:

a) $t(n,k)=0$ if $n<2k-1$;

b) $t(n,1)=n$ for any natural n.

Replacing $t(n,k)$ with $C^{k}_{n-(k-1)}$, we get:

1. $t(n,k)=C^{k}_{n-(k-1)}=0$ if $n-(k-1)<k$, hence, $n<2k-1$;

2. $t(n,1)=C^{1}_{n}=n$.

Thus, our hypothetical formula passes the test for the initial conditions.
Next, we examine it with the recursive formula

$$t(n,k)=t(n-1,k)=t(n-2,k-1).$$

We have:

$$t(n-1,k)+t(n-2,k-1)=C^{k}_{n-1-(k-1)}+C^{k-1}_{n-2-(k-2)}=$$

$$=C^{k}_{n-k}+C^{k-1}_{n-k}=C^{k}_{n-k+1}=C^{k}_{n-(k-1)}=t(n,k).$$

As we can see, the formula passes this test successfully as well. This completes the proof.

Problem 5.20. *How many ways are there to choose k numbers from*

$$1,2,3,\dots,n-1,n,$$

so that they differ from each other at least by 3*? Under which conditions there is at least one way to make such a choice?*

<u>*Answer.*</u> $C^{k}_{n-2(k-1)}$; $n\geq 3k-2$.

Problem 5.21. *(Generalization of two previous problems) How many ways are there to choose k numbers from*

$$1,2,3,\dots,n-1,n,$$

so that they differ from each other at least by s? Under which conditions there is at least one way to make such a choice?

<u>*Answer.*</u> $C^{k}_{n-(s-1)(k-1)}$; $n\geq sk-(s-1)$.

Hint. Establish a bijection between the sought subsets of the set $\{1, 2, 3, ..., n-1, n\}$ and all k-subsets of the set $\{1, 2, 3, ..., n-(s-1)(k-1)\}$. Follow the ideas of the solution to Problem 5.19.

Problem 5.22. *The numbers* $1, 2, 3, ..., n$ *are written in their usual order in the form of a circle (the number n is followed by* 1, *then goes* 2, *and so on). Every number has two neighbors: the neighbors of* 1 *are n and* 2; *the neighbors of* 2 *are* 1 *and* 3; *and so on; finally the number n has* $n-1$ *and* 1 *as its neighbors. How many ways are there to choose k of these numbers so that there are no neighbors among them?*

Answer. $\frac{n}{n-k} \cdot C^k_{n-k}$.

Solution. First Approach. This problem is the "round" analog of problem 19. Therefore, we can try using the result of that problem. Both problems consider k-element subsets of the set $\{1, 2, 3, ..., n\}$. We are not interested in all of them, but only in those which satisfy certain additional conditions. However, in both cases, the conditions are similar. We have to clarify the extent to which they are similar and define the differences. All neighboring numbers in the setting of problem 19 are also neighbors in the context of Problem 5.22. Here are the pairs of neighboring numbers: 1 and 2, 2 and 3, 3 and 4, and so on; finally, $n-1$ and n. There are no other neighbors according to the statement of Problem 5.19. In turn, Problem 5.22 defines one additional pair of neighbors, namely, n and 1. We can conclude, in order to get all those k-element subsets of the set $\{1, 2, 3, ..., n\}$ which we are required to count in Problem 5.22, we need remove those k-element subsets from Problem 5.19 which contain both 1 and n. According to the notation introduced above, there are $t(n, k)$ k-element subsets defined in Problem 5.19. How many of them include both numbers 1 and n? If a set contains the number 1, then it does not include the number 2. Similarly, if a set contains the number n, then there is no number $n-1$ in it. Thus, the rest of its elements (there are $k-2$ of them) should be chosen among the numbers $3, 4, ..., n-2$ with adherence to stated conditions. There are $t(n-4, k-2)$ ways to make this choice. If $\tau(n, k)$ denotes the number in question, then as a result of previous considerations we have

$$\tau(n, k) = t(n, k) - t(n-4, k-2)$$

We have already found that

$$t(n, k) = C^k_{n-(k-1)}.$$

To derive a similar formula for $\tau(n, k)$, it suffices to perform some arithmetical transforms. We have:

$$\tau(n, k) = t(n, k) - t(n-4, k-2) = C^k_{n-(k-1)} - C^{k-2}_{n-4-(k-3)} =$$

$$= C^k_{n-k+1} - C^{k-2}_{n-k-1} = \frac{(n-k+1)!}{k!(n-2k+1)!} - \frac{(n-k-1)!}{(k-2)!(n-2k+1)!} =$$

$$= \frac{(n-k-1)!}{(k-2)!(n-2k+1)!} \cdot \left(\frac{(n-k)(n-k+1)}{(k-1)k} - 1\right) =$$

$$= \frac{(n-k-1)!}{(k-2)!(n-2k+1)!} \cdot \frac{(n-2k+1)n}{(k-1)k} = \frac{(n-k-1)!}{k!(n-2k)!} \cdot \frac{(n-k)n}{(n-k)} =$$

$$= \frac{(n-k)!}{k!(n-2k)!} \cdot \frac{n}{n-k} = \frac{n}{n-k} C^k_{n-k}.$$

Second Approach. Split the k-element subsets in question into two classes: the subsets that include the number 1 and those that do not. If a subset contains the number 1, then it does not contain the numbers 2 and n. Therefore, other its elements form a $(k-1)$-element subset $\{3, 4, ..., n-1\}$, which besides does not contain adjacent numbers. There are $t(n-3, k-1)$ such subsets. Alternatively, if a subset does not include the number 1, then it is a k-element subset of the set $\{2, 3, ..., n\}$, and on top of this, it does not contain neighboring numbers. There are $t(n-1, k)$ such subsets. Thus, we have:

$$\tau(n, k) = t(n3, k-1) + t(n-1, k).$$

Applying the equality

$$t(u, v) = C_{u-v+1}^{v},$$

obtained in Problem 5.19, express the right-hand side of the above equality with the binomial coefficients:

$$\tau(n, k) = t(n-3, k-1) + t(n-1, k) = C_{(n-3)-(k-2)}^{k-1} + C_{(n-1)-(k-1)}^{k} =$$

$$= C_{n-k-1}^{k-1} + C_{n-k}^{k} = \frac{(n-k-1)!}{(k-1)!(n-2k)!} + \frac{(n-k)!}{k!(n-2k)!} =$$

$$= \frac{(n-k)!}{k!(n-2k)!}\left(\frac{k}{n-k} + 1\right) = \frac{n}{n-k} \cdot C_{n-k}^{k}.$$

Third Approach. Assume we have constructed various pairs $< A; s >$, where the first component A is a k-element subset of the set $\{1, 2, 3, ..., n\}$, which does not contain adjacent numbers (in the context of the current problem), and the second component s is a number from the set $\{1, 2, 3, ..., n\}$ which does not belong to A. How many such pairs exist? There are two different ways to construct such a pair. We can begin with the choice of the subset A, and then choose the number s from the remaining numbers. Alternatively, we can choose the number s first, and then construct the subset A out of other numbers. Depending on the approach to the construction of pairs $< A; s >$, there are different algorithms for counting such pairs.

Assume that we decided to construct pairs beginning with the choice of the subset A. Then the number of pairs can be counted as follows. There are $\tau(n, k)$ (we use the introduced above notation here, although the computational formula is still unknown) subsets A in total. Disregarding the exact choice of a subset, we will have $n-k$ options for the second component (any number s that is not included in A). Hence, $\tau(n, k) \cdot (n-k)$ pairs can be created (by virtue of the combinatorial rule of product).

Now, let us construct pairs beginning with their second components. As this component is a number from the set $\{1, 2, 3, ..., n\}$, there are n options for it. Assume we have exercised one of the options and have chosen some number to be s. How many ways to create the first component are there now? We will have to construct the subset A out of $n-1$ numbers which from the sequence

$$s+1, s+2, ..., n-1, n, 1, 2, 3, ..., s-1.$$

Constructing the subset A, we have to ensure that it does not contain any adjacent numbers (in the context of problem 19). Hence, the subset A can be constructed in $t(n-1, k)$ ways.

As we can see, this number does not depend on the exact value of s chosen to be the second component of the pair. We conclude that by virtue of the rule of product, there are $n \cdot t(n-1, k)$ pairs $< A; s >$.

Comparing the results of different approaches to the construction of pairs $< A; s >$, we get the equation

$$\tau(n, k) \cdot (n-k) = n \cdot t(n-1, k)$$

for the sought value of $\tau(n, k)$.

Taking into account that

$$t(n-1, k) = C_{n-k}^{k}$$

(see Problem 5.19, the above equation yields the direct computational formula for $\tau(n, k)$:

$$\tau(n, k) = \frac{n}{n-k} C_{n-k}^{k}.$$

Problem 5.23. *Consider the permutations of the set*

$$N_n = \{1, 2, 3, \ldots, n\}.$$

Introduce the following notation:

Π_{11} *is the set of all permutations which have the number* 1 *in the first position;*
Π_{12} *is the set of all permutations which have the number* 1 *in the second position;*
Π_{22} *is the set of all permutations which have the number* 2 *in the second position;*
Π_{23} *is the set of all permutations which have the number* 2 *in the third position;*
and so on;
Π_{ii} *is the set of all permutations which have the number i in the i-th position;*
$\Pi_{i,i+1}$ *is the set of all permutations which have the number i in the $i+1$-th position;*
and so on; finally,
Π_{nn} *is the set of all permutations which have the number n in the n-th position;*
Π_{n1} *is the set of all permutations which have the number n in the first position;*

1. *How many elements (permutations) are there in each of $2n$ sets Π_{ij} $(i = 1, 2, ..., n;$ $j = 1, 2, ..., n)$?*

2. *Let the sets Π_{ij} be placed around a circle in the following order:*

$$\Pi_{11}, \Pi_{12}, \Pi_{22}, \Pi_{23}, \Pi_{33}, \Pi_{34}, ..., \Pi_{n-1,n}, \Pi_{nn}, \Pi_{n1}$$

   (Π_{n1} is followed by Π_{11} and so on).

   Prove that any two adjacent sets from this circle of sets do not have common elements, that is $\Pi_{ij} \cap \Pi_{js} = \emptyset$ for any i, j, s.

   Prove that if Π_{ij} and Π_{pq} are not adjacent, then $\left|\Pi_{ij} \cap \Pi_{pq}\right| = (n-2)!$

3. *Let $\Pi_{i_1 j_1}, \Pi_{i_2 j_2}, ..., \Pi_{i_k j_k}$ be k different sets from our circle and there are no neighbors among them. Prove that*

$$\left|\Pi_{i_1 j_1} \cap \Pi_{i_2 j_2} \cap \ldots \cap \Pi_{i_k j_k}\right| = (n-k)!$$

4. *Basing on the results of three previous sections and the inclusion-exclusion principle, determine the amount of elements (permutations) of the union of the sets Π_{ij}, which is*

$$\Pi_{11} \cup \Pi_{12} \cup \Pi_{22} \cup \Pi_{23} \cup \ldots \cup \Pi_{n-1,n} \cup \Pi_{nn} \cup \Pi_{n1}.$$

5. *How many permutations of the numbers* $1, 2, 3, \ldots, n$ *possess all following properties: the number 1 does not stand in any of the first two positions; the number* 2 *does not stand in the second or third positions; the number* 3 *does not stand in the third or fourth positions; and so on; finally, the number n does not stand in n-th (the last) or the first positions?*

Answer.
1) $(n-1)!$;
4) $\tau(2n,1)(n-1)! - \tau(2n,2)\cdot(n-2)! + \tau(2n,3)\cdot(n-3)! - \tau(2n,4)\times$
$\times(n-4)! + \ldots + (-1)^{k+1}\tau(2n,k)\cdot(n-k)! + \ldots + (-1)^{n+1}\tau(2n,n)\cdot 0!$,

$$\tau(2n,k) = \frac{2n}{2n-k} C^k_{2n-k}.$$

5) $n! - \tau(2n,1)\cdot(n-1)! + \tau(2n,2)\cdot(n-2)! - \ldots + (-1)^n\tau(2n,n)\cdot 0!$

Solution. 1) The set Π_{ij} consists of those permutations of the numbers $1, 2, 3, \ldots, n$ in which the number i occupies the j-th place. Other $n-1$ numbers can be in any of $n-1$ places. There are $(n-1)!$ permutations of this type.

2) Two neighboring sets of permutations are either of the form Π_{ii}, Π_{ij} $(i \neq j)$, or Π_{ij}, Π_{jj} $(i \neq j)$. Consider both cases. If a permutation belongs to the set Π_{ii}, then it has the number i in i-th place. If a permutation belongs to the set Π_{ij}, then the number i stands in j-th place in it. This means that the same permutation can not belong to both sets Π_{ii} and Π_{ij} at the same time, and hence, to their intersection, because the number i can not be in the i-th and the j-th places at the same time (as $i \neq j$).

Let us analyze the second case. Let a permutation belongs to the set Π_{ij}. Then there is the number i in its j-th place. If this permutation belonged to the set Π_{jj} as well, then there would have been the number j in the j-th position. These two events are incompatible: two different numbers (as $i \neq j$) can not occupy the same position in a permutation. We can conclude that any two adjacent sets of our circle do not share any elements (permutations).

Now, let Π_{ij} and Π_{pq} be not adjacent sets. Then $i \neq p$ and $j \neq q$. All permutations of the set Π_{ij} have the number i in their j-th places. There are no other restrictions concerning the permutations in this set. Similarly, all permutations of the set Π_{pq} have the number p in their q-th places, and this is the only and the defining (characteristic) property of this set of permutations. As $i \neq p$ and $j \neq q$, these two properties are compatible: there exist permutations that have the number i in the j-th place and the number p in the q-th place. Such permutations are the only permutations that form the intersection $\Pi_{ij} \cap \Pi_{pq}$. There are $(n-2)!$ of them, as a result of the combinatorial rule of product. Indeed, in order to create such permutation, on needs to place $n-2$ different numbers (namely, integers from the interval $[1,n]$, except for i and p) in $n-2$ different positions (except for the j-th and the

q-th). This is the same as creating a permutation of $n-2$ numbers, and there are $(n-2)!$ ways to perform this.

3) The set $\Pi_{i_1 j_1} \cap \Pi_{i_2 j_2} \cap \ldots \cap \Pi_{i_k j_k}$ consists of all those permutations, in which the numbers $i_1, i_2, \ldots, i_k$ stand in the $j_1, j_2, \ldots, j_k$-th places respectively (the numbers of places j_1, j_2, ..., j_k are all different, as well as the numbers i_1, i_2, ..., i_k because there are no adjacent among the considered sets). The rest $n-k$ places are occupied by the remaining $n-k$ numbers. The placement of these $n-k$ numbers is the factor which distinguishes one permutation from others. And there are $(n-k)!$ ways to place $n-k$ different numbers in $n-k$ positions.

4) Let us express the inclusion-exclusion formula for $2n$ sets Π_{ij} as follows:
(The number of permutations in the union of the sets Π_{ij})=
=(Sum of the first level block numbers) –
–(Sum of the second level block numbers)+
+(Sum of the third level block numbers) –
–. . . . . +
+ $(-1)^{k+1}$ (Sum of the k-th level block numbers)+
+. . . . . +
+ $(-1)^{n+1}$ (Sum of the n-th level block numbers).

In three previous paragraphs, we have determined the block numbers of all levels. All first level block numbers are the same and equal to $(n-1)!$ (paragraph 1). As to the second level block numbers, part of them are zero and other are equal to $(n-2)!$. $(n-2)!$ is the values of those block numbers that correspond to the intersections of isolated (not neighboring) sets of Π_{ij}). The number of such intersections was considered in Problem 5.22, where we denoted it by $\tau(2n, n)$. Thus, the second pair of parentheses in the right-hand side of the inclusion-exclusion formula equals to $\tau(2n, n) \cdot (n-2)!$. Following this scheme (according to the results of paragraph 3 and Problem 5.22, we can evaluate the sums of block numbers of all further levels. Sum of k-th level block numbers $(k = 1, 2, 3, \ldots, n)$ equals $\tau(2n, n) \cdot (n-k)!$. Therefore, the inclusion-exclusion formula for the union of the sets Π_{ij}) $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$ becomes:

$$|\Pi_{11} \cup \Pi_{12} \cup \Pi_{22} \cup \Pi_{23} \cup \ldots \cup \Pi_{n-1,n} \cup \Pi_{nn} \cup \Pi_{n1}| =$$

$$= \tau(2n, 1) \cdot (n-1)! - \tau(2n, 2) \cdot (n-2)! + \tau(2n, 3) \cdot (n-3)! - \ldots$$

$$\ldots + (-1)^{k+1}\tau(2n, k) \cdot (n-k)! + \ldots + (-1)^{n+1}\tau(2n, n) \cdot 0!$$

It remains to recall the formula for $\tau(2n, k)$, which has been derived in Problem 5.22:

$$\tau(2n, k) = \frac{2n}{2n-k} \cdot C_{2n-k}^{k}.$$

5) The permutations in question are all permutations that fall outside the union of the sets Π_{ij}). Hence, there are

$$n! - \tau(2n, 1) \cdot (n-1)! + \tau(2n, 2) \cdot (n-2)! - \tau(2n, 3) \cdot (n-3)! +$$
$$+ \ldots + (-1)^k \tau(2n, k) \cdot (n-k)! + \ldots (-1)^n \tau(2n, n) \cdot 0!$$

of them in total.

Chapter 6

Trajectories Inside a Circle

In this chapter, we consider some extensions to Problem 1.52 from the combinatorial rule of product section. Below, several problems concerning polygonal chains with vertices on a circle are solved. These problems indicate that even considering a rather narrow geometric topic, it is possible to immerse oneself in problematic and methods of combinatorial calculations. calculations.

1. Zigzags in a Circle without Self-Intersections

Let us take n $(n \geq 2)$ points on a circle. For our convenience, we further call them base points. Choose one of them to be the initial point and denote it by A. The rest of the base points are denoted with the letters B, C, D, and so on.

Example 6.1. *Imagine that one needs to depart from the initial point A and visit each of base points once, moving along chords of the circle. The journey ends in one of the base points. How many ways are there to perform this if the whole trajectory should consist of the chords connecting the base points, which do not intersect inside the circle?*

For example, if there are 4 base points on a circle $(n = 4)$, namely A, B, C, D (see Fig.6.1), then there are four wanted paths (trajectories): $ABCD, ABDC, ACBD$ and $ACDB$.

Let us get down to the solution. Let B and C be the base points adjacent to the initial point A. This means that one of two arcs in which each of the points B and C split the circle into, there is no other base point except for A. A polygonal chain which is the subject of the question, can not begin with any line segment other than AB orAC, as any other chord AP splits the points B and C, so after visiting one of them we inevitably cross the chord in order to visit the other. Hence, any polygonal chain should begin with one of the segments AB or AC. These two cases are equivalent (or symmetrical) in the sense that it is impossible to imagine why there could be more polygonal chains beginning with the line segment AB than those beginning with AC. If we denote the wanted amount by $l(n)$ (we remind that n is the number of the base points), then

$$l(n) = 2 \cdot l_{AB}(n),$$

where $l_{AB}(n)$ is the amount of polygonal chains beginning with the line segment AB. Subsequent line segments of every such polygonal chain form a polygonal chain with the starting

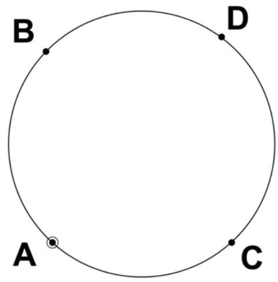

Figure 6.1. Zigzags in a circle without self-intersections.

point B, which includes all base points, except for A, and possess all the required properties. This means that

$$l_{AB}(n) = l(n-1),$$

hence,

$$l(n) = 2 \cdot l(n-1).$$

This formula is the decisive one. It remains to highlight that it is correct for any values of n greater than 2. Thus, we can construct the following chain of equalities:

$$\begin{aligned}
&l(n) = 2 \cdot l(n-1),\\
&l(n-1) = 2 \cdot l(n-2),\\
&l(n-2) = 2 \cdot l(n-3),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&l(5) = 2 \cdot l(4),\\
&l(4) = 2 \cdot l(3),\\
&l(3) = 2 \cdot l(2).
\end{aligned}$$

There are $n-2$ equalities in total. Multiplying them with each other term-wise (the right-hand sides and the left-hand sides separately) and then dividing the resulting equality by

$$l(n-1) \cdot l(n-2) \cdots l(4) \cdot l(3),$$

we get

$$l(n) = 2^{n-2} \cdot l(2).$$

Undoubtedly, $l(2) = 1$, hence,

$$l(n) = 2^{n-2}.$$

Example 6.2. *How many polygonal chains with the properties described in the previous problem have arbitrary (but two different) base points as their ends?*

In other words, we do not consider the notion of the initial point in this problem. Instead, we are dealing with polygonal chains which connect two different base points and include every other base point once. The restriction that polygonal chains should not have self-intersections is still available.

Solution. According to the previous problem, there are 2^{n-2} polygonal chains, which have a fixed end A. The role of the point A might as well be played by any other base point, and there are n of them overall. However, the product $2^{n-2} \cdot n$ is not the wanted number, because any polygonal chain has two ends, each of which can be considered as initial. Multiplying 2^{n-2} by n, we account for each polygonal chain twice. So in fact, there are

$$2^{n-3} \cdot n$$

polygonal chains of interest.

Example 6.3. *Let there be $2+n+m$ base points on a circle, namely: the points A and B, n points on the "left" arc AB and m points on the "right" arc AB. How many polygonal chains without self-intersections have their ends in the points A and B and vertices in other $n+m$ base points (and only in them), with every such point being the ending point of exactly two segments of the chain?*

As we can see, this time we are talking about the number of ways to travel from the point A to the point B, moving along the chords that connect the base points under two additional restrictions: every base point should be visited exactly once and the polygonal chain should have no self-intersections.

Solution. First, we introduce convenient notation. Moving from A to B along the left arc, denote the base points by $C_1, C_2, C_3, \ldots, C_n$. Similarly, the base points on the right arc AB will be denoted by $D_1, D_2, D_3, \ldots, D_m$ (D_1 is the closest base point to the point A, D_2 is the next, and so on).

Assume that we construct the required polygonal chain departing with its ending point A (we can call it the origin of the chain). To define one of such polygonal chains is to provide the ordered list of its consecutive vertices, the last of which would be B and the transitional points are all base points C_i and D_j $(i = 1, 2, 3, \ldots, n;\ j = 1, 2, 3, \ldots, m)$ appearing one time each. A polygonal chain has no self-intersections if and only if in this list the point C_1 appears before the point C_2, C_2 before C_3, and so on. Similarly, the points $D_1, D_2, D_3, \ldots, D_m$ should appear in the order of increase of their indices rather than be shuffled somehow. In addition, the points C_i can randomly alternate with the points D_j. For instance, we provide the complete count of such lists for the case $n = 2$, $m = 3$:

1. $AC_1C_2D_1D_2D_3B$
2. $AC_1D_1C_2D_2D_3B$
3. $AC_1D_1D_2C_2D_3B$
4. $AC_1D_1D_2D_3C_2B$
5. $AD_1C_1C_2D_2D_3B$
6. $AD_1C_1D_2C_2D_3B$
7. $AD_1C_1D_2D_3C_2B$
8. $AD_1D_2C_1C_2D_3B$

9. $AD_1D_2C_1D_3C_2B$

10. $AD_1D_2D_3C_1C_2B$.

Why are there exactly 10 lists? Obviously, we need an explanation that would fit the case of arbitrary n and m. Consider any of the above ten lists, say, the seventh. If we remove indices accompanying the letters C_i and D_j, then we get the following sequence of letters A, B, C and D:

$$ADCDDCB. \tag{6.1}$$

It can be considered as the code of list 7) and the corresponding polygonal chain. Having a polygonal chain, we can easily construct list 7), and then find its code. All we have to do is just drop the indices when we write down the letters of a list. Conversely, given the code (6.1), it is straightforward to recover the corresponding list, as the indices of the letters C and D (separately) form an increasing sequence of consecutive natural numbers. Therefore, there is a bijection between the code and all polygonal chains that connect A and B with no self-intersections.

So, what is the code of a polygonal chain in the general case, where there are n base points C_i on the arc AB and m base points D_j on the other arc? This is a certain sequence of letters A, B, C and D. It begins with A and ends with B. There are n letters C and m letters D between them. Thus, the length of the entire code (the number of letters in it) is $n+m+2$. The initial and the last letters of all codes are the same (A and B), hence, they do not affect their lengths. It (the length of a code) only depends on the number of ways to place n letters C and m letters D in a straight line. In order to find this number, we can think as follows (though this is not the only possible algorithm). There are $n+m$ positions (squares, cells) in a line: the first, second, third, and so on up to the last one which has the number $n+m$. How many ways are there to place n letters C and m letters D in these positions (fill in the cells with them)? In order to realize one of many possible options, it is sufficient to choose the positions for the letter C. The remaining positions are filled in with the letter D automatically. Hence, the problem has been reduced to the choice of n positions of $n+m$ available. As we know, there are C_{n+m}^{n} ways to make this choice. Therefore, the amount of wanted polygonal chains is the same. For instance, for $n=2$, $m=3$ there are $C_{2+3}^{2} = C_5^2 = \frac{5\cdot 4}{1\cdot 2} = 10$ codes (polygonal chains).

2. Trajectories in a Circle with Self-Intersections

Example 6.4. *Consider n points on a circle, which we further call base points. Let us denote one of them by the letter A and assume it to be the initial point. Departing from the point A, we are about to move along the chords that connect the base points. We want to visit every base point once and finish our journey in the last of them. Thus, our path is a polygonal chain consisting of $n-1$ line segments. One of its ends (start) is the point A, and the other is some other base point. Finally, the last and the most important condition: we should move along those polygonal chains which have exactly one self-intersection (see Fig. 6.2, where the case of seven base points is presented). How many different ways are there to make such a trip?*

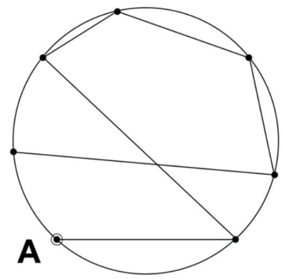

Figure 6.2. Trajectories in a circle with self-intersections.(a)

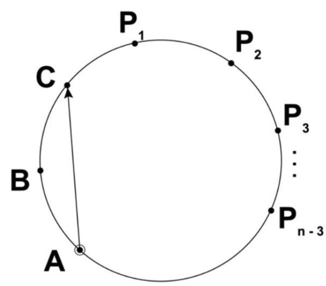

Figure 6.3. Trajectories in a circle with self-intersections.(b)

Solution. Denote the wanted number by $u(n)$. Let P and Q be base points adjacent to A. If we begin our journey by visiting P or Q, then there are $u(n-1)$ ways to extend our path in both cases. Hence,

$$u(n) = 2u(n-1) + v \tag{6.2}$$

The value of the summand v is yet to be determined. It denotes the amount of all those trajectories that begin with a chord that connects A with one of the remote (not adjacent) base points.

First, consider the case where the first line segment of a polygonal chain is AC (see Fig. 6.3), and there is only one base point in one of two arcs AC. Given $n \geq 5$, there are at least 2 base points on the other arc. In this case, there are 3 polygonal chains:

$$ACBP_1P_2...P_{n-3};$$
$$ACBP_{n-3}P_{n-4}...P_1;$$
$$ACP_1P_2...P_{n-3}B.$$

Now, consider the case where there are at least two base points on each of two arcs AC.

In Fig. 6.7, they are denoted by $S_1, S_2, ..., S_k$ and $T_1, T_2, ..., T_r$ $(k \geq 2,\ r \geq 2,\ k+r = n-2)$. This time, we have 4 polygonal chains with one self-intersection, which begin with the line segment AC, namely:

$$ACS_1S_2...S_kT_1T_2...T_r,$$

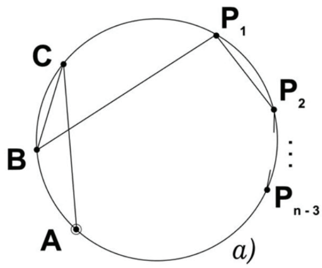

Figure 6.4. Trajectories in a circle with self-intersections.(c)

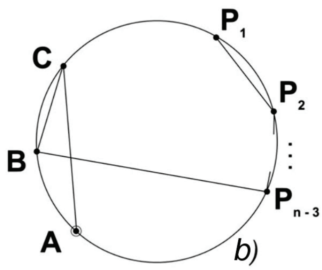

Figure 6.5. Trajectories in a circle with self-intersections.(d)

$$ACS_1S_2...S_kT_rT_{r-1}...T_1,$$
$$ACT_1T_2...T_rS_1S_2...S_k,$$
$$ACT_1T_2...T_rS_kS_{k-1}...S_1.$$

Taking into account that there are $n-3$ options for the position of the initial point, for two of which there is exactly one base point on one of two arcs AC, we can determine the value of the summand v in formula (6.2). It is equal to:

$$v = 2\cdot 3 + (n-5)\cdot 4 = 4n-14.$$

Thus, formula (6.2) transforms into

$$u(n) = 2u(n-1) + 4n - 14. \tag{6.3}$$

We can easily ensure that for $n = 4$, there exist only two polygonal chains with the initial point A and one self-intersection.

They are shown on Fig. 6.8. For greater values of n, recurrence relation (6.3) comes into play. Applying it for the values $n = 5, 6, 7, 8, 9, ...$, step by step, we can (at least, theoretically) reach any given natural number and find the answer to the question about the number of polygonal chains in that case. In this sense, we can consider the problem solved. The answer is:

$$u(4) = 2, \quad n < 4, \quad u(n) = 2u(n-1) + 4n - 14, \quad n \geq 4. \tag{6.4}$$

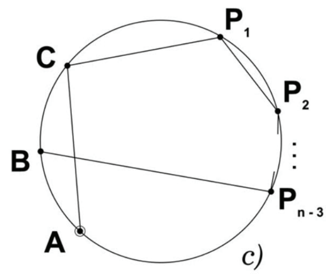

Figure 6.6. Trajectories in a circle with self-intersections.(e)

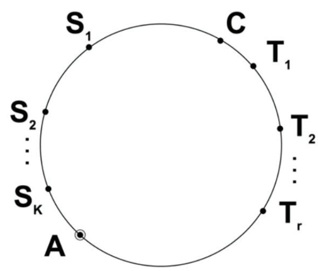

Figure 6.7. Trajectories in a circle with self-intersections.(f)

There is an alternative meaning of this result. Our original task was to find the computational formula for the function $u(n)$ defined on the set of natural numbers $4, 5, 6, 7$, and so on. If we write the values of this function in the order of increase of the argument n, then we get some numeric sequence. If we know the rule that allows writing down all elements of a sequence from its beginning to (potentially) infinity, then we have undeniable grounds to claim that we know that sequence. The information expressed by two equalities c is exactly this type of rule. Below in table 6.1, we present the beginning of our sequence (in the first row, there are consecutive values of n, and in the second, there are the elements of the sequence):

Table 6.1. Values of the function $u(n)$

n	4	5	6	7	8	9	10	11
$u(n)$	2	10	30	74	166	354	734	1498

As we know, formulas of the type of the second equality in (6.4) (or equality (6.3)) are called recursive. If we ask formula (6.3), say, what is $u(9)$ equal to, then the “reply” would be: find the value of $u(8)$ first; then find the number $u(9)$ from the equality:

$$u(9) = 2 \cdot u(8) + 4 \cdot 9 - 14.$$

Formula (6.3) only starts operating efficiently after we somehow, using independent from its

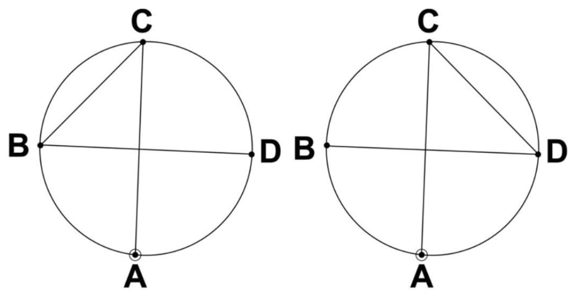

Figure 6.8. Trajectories in a circle with self-intersections.(g)

technique, calculate one value of the function $u(n)$. After that, similarly to the skyscraper building upon its foundation, formula (6.3) will produce the values of this function for greater values of n one after another. In our case, it is straightforward to ensure that $u(4) = 2$. The following numbers $u(5), u(6), u(7)$, and so on up to infinity can be written basing on the recurrence relation (6.3), having no regard to their relation to the original problem.

Thus, having found formulas (6.4), which uniquely define the sequence $u(n)$, we may consider the problem solved.

However, it is very desirable to find the answer to the problem in a slightly different form, if it is possible. We are talking about a formula that could express the values of the function $u(n)$ directly through the values of n. Thus, one could input the wanted value of n into a formula, perform the required calculations and get the wanted number $u(n)$. For instance, in order to calculate $u(195)$ with the help of such a formula, we do not need the values $u(194), u(193)$, etc. The examples of direct functions which explicitly and with no intermediaries establish the connection between the values of n and functions depending on it are:

$$2n+n+5,\ 2\cdot 3^n+n,\ \frac{1}{3}n(n^2-1),\dots$$

So, is there a way to find the direct formula that allows calculating every number of a sequence by its number, when we are given recurrence relation (R) with the initial condition? This is not a trivial task. Moreover, often it is even impossible to complete. Obviously, the complexity of this task essentially depends on the type of recursive formula. In addition, this is not an algorithmic task: there is no universal approach to perform the move from a recurrence relation to a direct formula even in the cases when such a move is possible. Therefore, every time one encounters such a task it results in solving a serious, often elegant, and instructive mathematical problem.

Let us try deducing the direct formula for $u(n)$ in our problem.

Below we present one of the varieties of appropriate methods. Thorough analysis of recurrence relation (6.3) reveals that if we remove two last summands ($4n$ and -14), then it transforms into the formula of the well-known type

$$u(n) = 2u(n-1).$$

Thus, we would have got the geometric progression. The situation would get even more straightforward if there were no first summand in the right-hand side of (6.3). In this

case, the formula becomes the direct one $(u(n) = 4n - 14)$, and in addition, it defines the renowned sequence: the arithmetic progression with difference 4.

These two features of the formula (6.3) suggest a hypothesis: could it be that the sequence $u(n)$ is the sum of the geometric and arithmetic progressions? In particular, this hypothesis is attractive because it is easily verifiable. If it appears to be true, then it will be a complete success for us, and if not, there will be no regret for the time wasted.

According to the hypothesis, the direct formula for $u(n)$ is as follows:

$$u(n) = a \cdot 2^n + bn + c \tag{6.5}$$

(clearly, it is natural to assume that the common ratio of the geometric progression is 2). Here, a, b and c are unknown. If our guess about the direct formula for $u(n)$ is correct, how can these numbers be determined? First, for $n = 4$ the formula should produce the value of 2 (the initial condition of recurrence relation (R)). Hence, for the unknowns a, b and c, we have the equation

$$16a + 4b + c = 2.$$

For $n \geq 5$, hypothetical formula (6.5) has to satisfy recurrence relation (6.3), that is the equality

$$a \cdot 2^n + bn + c = 2(a \cdot 2^{n-1} + b(n-1) + c) + 4n - 14$$

should hold for any integer $n \geq 5$. Elementary transforms yield the equality

$$(b+4) \cdot n + (c - 2b - 14) = 0.$$

The left-hand side is the linear function of n. It can be equal to zero only if both its coefficients are zero. Recalling the equation derived above, we get the following system of equations concerning the unknowns a, b and c (three equations with three unknowns):

$$\begin{cases} 16a + 4b + c & = & 2, \\ b + 4 & = & 0, \\ c - 2b - 14 & = & 0. \end{cases} \tag{6.6}$$

The fate of our hypothesis is in the "hands" of this system. If there is a solution to it, then we are jubilant. Otherwise, we will have to reconcile with a temporary and minor failure and look for other ways to succeed.

The great news is that system (6.6) does have a solution (besides, as it comes from the properties of recurrence relation (6.4), there could be no more than one solution to this system). From the first, second and third equations, we consecutively find that $b = -4$, $c = 6$, $a = \frac{3}{4}$. So the direct formula for $u(n)$ is

$$u(n) = 3 \cdot 2^{n-2} - 4n + 6. \tag{6.7}$$

Both to check ourselves and for our pleasure, we will further apply the above formula to calculate several values of $u(n)$ for small values of n and compare them to the ones derived by the recursive formula.

Direct formula (6.7) can be derived using similar but slightly different algorithm. Taking into account the properties of recurrence relation (R) that has been established above,

it is appropriate (or, say, acceptable) to attempt correcting the function $u(n)$ with a linear with respect to n summand so that it becomes a geometric progression (of course, with the common ratio of 2):

$$v(n) = u(n) + xn + y,$$

$v(n) = 2v(n-1)$ for $n \geq 5$.

Considering c, the above yields:

$$\begin{array}{rcl} u(n) = x \cdot n + y & = & 2(u(n-1) + x \cdot (n-1) + y), \\ 2u(n-1) + 4n - 14 + xn + y & = & 2u(n-1) + 2x \cdot (n-1) + 2y, \\ (4-x) \cdot n + (2x - y - 14) & = & 0. \end{array}$$

The last equality should hold irrespective of the values of n, hence,

$$\begin{cases} 4 - x = 0, \\ 2x - y - 14 = 0, \end{cases}$$

which gives $x = 4,\ y = -6$.

Thus,

$$v(n) = u(n) + 4n - 6. \tag{6.8}$$

In addition, we have

$$v(4) = u(4) + 4 \cdot 4 - 6 = 2 + 16 - 6 = 12.$$

Now, one might recall the well-known formula for arbitrary element of a geometric progression or act as follows:

$$\begin{array}{l} v(4) = 12, \\ v(5) = 2 \cdot v(4), \\ v(6) = 2 \cdot v(5), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ v(n) = 2v(n-1). \end{array}$$

Multiplying the above equalities term-wise (the left-hand sides and the right-hand sides separately) and removing similar factors from both sides of the resulting equality, we get

$$v(n) = 12 \cdot 2^{n-4} = 3 \cdot 2^{n-2}.$$

The formula (6.7) now follows from equality (6.8).

Let us return to the moment when we have found recurrence relation (6.4) and calculated the initial values of the function $u(n)$. Is there a chance that we could find the direct formula (6.7) without guess about its form (the sum of power and linear functions)? Are there any other ways to get (6.7) from (6.4)?

It often appears that great results come as a result of a thorough analysis of initial terms of the sequence aimed at the discovery of patterns that are not explicit at the first glance at a recurrence relation. Line up several initial terms of the sequence, which we have determined above:

$2, 10, 30, 74, 166, 354, 734, 1498, \ldots$

Observing these numbers does not provide much optimism. Instead, let us consider the sequence of differences of the neighboring numbers of the above sequence:

$8, 20, 44, 92, 188, 380, 764, \ldots$

Again, we can see nothing helpful. However, if we decide to construct the sequence of difference of numbers of the latter sequence, then we will come across something really encouraging:

$12, 24, 48, 96, 192, 384, \ldots$

A little experience is required to realize that the latter sequence is the geometric progression with the common ratio of 2. We immediately investigate how do its elements (second differences) relate to the elements of the original sequence. The first differences are

$$\begin{array}{ccccc} u(5)-u(4) & u(6)-u(5) & u(7)-u(6) & u(8)-u(7) & u(9)-u(8) \\ 8 & 20 & 44 & 92 & 188 \end{array}$$

For the purpose of convenience, express the differences as a column:

$$\left.\begin{array}{l} u(6)-2u(5)+u(4)=12, \\ u(7)-2u(6)+u(5)=24, \\ u(8)-2u(7)+u(6)=48, \\ u(9)-2u(8)+u(7)=96, \\ u(10)-2u(9)+u(8)=192, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right. \quad \text{(K beginning)}$$

Assuming that the observed pattern of the differences (each next difference is twice the previous one) is inherent to the whole sequence of such differences, we extend this column up to the moment when the equality begins with $u(n)$ arises. Here are the last four equalities of the column:

$$\begin{array}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u(n-3)-2u(n-1)+u(n-5)=12\cdot 2^{n-9}, \\ u(n-2)-2u(n-3)+u(n-4)=12\cdot 2^{n-8}, \quad \text{(K end)} \\ u(n-1)-2u(n-2)+u(n-3)=12\cdot 2^{n-7}, \\ u(n)-2u(n-1)+u(n-2)=12\cdot 2^{n-6}. \end{array}$$

Adding the equalities of the column (K) term-wise, we get:

$$\begin{array}{l} u(n)-u(n-1)-u(5)+u(4)= \\ =12(1+2+2^2+2^3+\cdots+2^{n-8}+2^{n-7}+2^{n-6})= \\ =12\cdot(2^{n-5}-1). \end{array}$$

Replacing $u(5)$ and $u(4)$ with their numeric values, we arrive at the equality:

$$u(n)-u(n-1)=3\cdot 2^{n-3}-4.$$

This is another recurrence relation again. Are we going through a vicious cycle here? Not at all! Even though the new recurrence relation is no better than the previous one, it is <u>different</u> and together they (the new one and the old) lead us to our goal. Removing $u(n-1)$ from the system of equalities

$$\begin{cases} u(n)-2u(n-1) &= 4n-14, \\ u(n)-u(n-1) &= 3\cdot 2^{n-3}-4 \end{cases}$$

we find the desired equality

$$u(n)=3\cdot 2^{n-2}-4n+6.$$

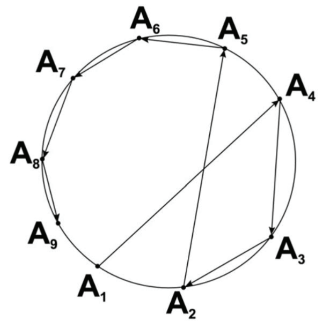

Figure 6.9. Polygonal chains. (a)

Example 6.5. *Suppose that $A_1, A_2, A_3, \ldots, A_n$ are consecutive points on a circle. How many polygonal chains with one self-intersection have vertices in the points $A_2, A_3, \ldots, A_{n-1}$ and ends in the points A_1 and A_n?*

Solution. Assume that A_1 is the initial point and A_n is the closing (ending) one of the polygonal chains. Denote the number of interest by $\alpha(n)$. Split all polygonal chains in question into two groups: those that begin with the line segment A_1A_2, and those that begin with another line segment A_1A_k $(k \neq 2)$. Obviously, the index k can attain values from 3 to $n-1$. Let us attempt to answer the question: how many polygonal chains are there in each of these two groups? First, agree that we can freely use the (symbolic) notation of the introduced function $\alpha(n)$, e.g., $\alpha(n-1)$, $\alpha(n-2)$, etc. Indeed, using these symbols, we have a chance to get a recurrence relation for $\alpha(n)$, which either will solve the problem in the best case or will be an intermediate step on our way to success.

If a polygonal chain begins with the line segment A_1A_2, then its extension up to the ending point A_n is a polygonal chain with one self-intersection, the opening point A_2, the ending point A_n and intermediary points $A_3, A_4, \ldots, A_{n-1}$. According to our notation, there are $\alpha(n-1)$ such polygonal chains. Therefore, there is the same amount of polygonal chains in our first group.

Imagine a polygonal chain that begins with the chord A_1A_k, where $3 \leq k \leq n-1$. What can be its next line segment? It is necessarily A_kA_{k-1}. Why? 1) the line segment A_kA_l $(l > k)$ can not be the second one, otherwise, the polygonal chain will have to visit, say, vertex A_{k-1} and get to the point A_n finally. Thus, the first line segment A_1A_k will be crossed twice by such a polygonal chain. 2) The line segment A_kA_s $(s < k-1)$ can not be the second one as well, because the polygonal chain will cross A_kA_s (as it should visit the vertex A_{k-1}) and A_1A_k (as it should end in the point A_n). Hence, the only allowed option is A_kA_{k-1}. Observing the following line segments of the polygonal chain, we ascertain that there are no options for the next direction on any step. Its path is predefined by the first step A_1A_k. Below we present the sequence of visiting of the points on a circle:

$$A_1A_kA_{k-1}A_{k-2}\cdots A_3A_2A_{k+1}A_{k+2}\cdots A_{n-2}A_{n-1}A_n. \tag{6.9}$$

In Fig. (6.9), the polygonal chain is drawn for $n = 9$, $k = 4$.

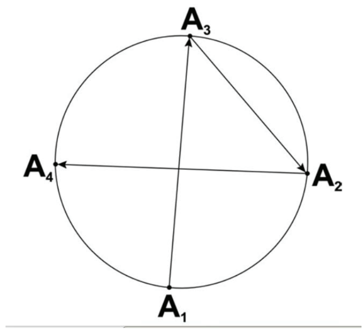

Figure 6.10. Polygonal chains. (b)

Thus, there is only one polygonal chain for any value of k from the interval from 3 to $n-1$. This means that the second group contains $n-3$ polygonal chains (same as the amount of possible values of k).

We have enough information to construct the recursive formula

$$\alpha(n) = \alpha(n-1) + n - 3. \tag{6.10}$$

It remains to accompany it with the initial condition

$$\alpha(4) = 1, \tag{6.11}$$

which is straightforward to verify: there is only one way to get from the point A_1 on a circle to the point A_4 along with three chords, two of which intersect (see Fig. 6.10).

Using the approach applied in several of the previous problems, we get the direct formula for $\alpha(n)$ without much effort (list equalities (6.11) and (6.10) in a column for all values of the parameter from 5 to n, and then sum up the resulting equalities term-wise):

$$\begin{array}{l}
\alpha(4) = 1, \\
\alpha(5) = \alpha(4) + 2, \\
\alpha(6) = \alpha(5) + 3, \\
\alpha(7) = \alpha(6) + 4, \\
\dots\dots\dots\dots\dots\dots \\
\alpha(n-2) = \alpha(n-3) + (n-5), \\
\alpha(n-1) = \alpha(n-2) + (n-4), \\
\alpha(n) = \alpha(n-1) + (n-3). \\
-------------- \\
\alpha(n) = 1 + 2 + 3 + 4 + \ldots + (n-5) + (n-4) + (n-3) = \\
= \frac{1+(n-3)}{2} \cdot (n-3) = \frac{(n-2)(n-3)}{2}.
\end{array}$$

The problem is solved. The answer is provided in the best possible form. Besides, the formula seems familiar. Very familiar, in fact. The half of the product of two consecutive natural numbers ... If we denote the greater of them by m, then the right-hand side gains more usual form $\frac{m(m-1)}{2}$

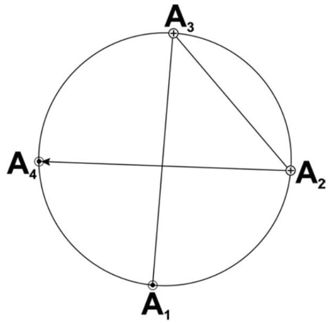

Figure 6.11. Correspondence between a pair of inner vertices of a polygonal chain and a polygonal chain with one self-intersection.

More usual, and thus more recognizable. Yes, this is the binomial coefficient C_m^2. Thus, we get

$$\alpha(n) = C_{n-2}^2.$$

This answer is given in the perfect form, which encourages us to continue considering the problem. One can hope that the same perfect answer can be obtained as a result of an elegant and brief solution. Let us recall the combinatorial sense of the symbol C_{n-2}^2. It denotes the number of ways to choose 2 objects out of available $n-2$. In other words, C_{n-2}^2 is the amount of two-element subsets of an $n-2$-element set.

And now, answering the question about the amount of polygonal chains of a certain type, we have received the very same number C_{n-2}^2. We conclude that there is a bijection between these polygonal chains and two-element subsets of an $(n-2)$-element set. If we are able to establish such a bijection directly, then we will get a really elegant solution, the style of which fits the answer best.

It is clear, which part of the problem contains the wanted $(n-2)$-element set. In fact, there is only one set in the statement of the problem. This is the set of base points on a circle $A_1, A_2, \ldots, A_n$. There are n such points in total, and the first and last of them play the same role in the construction of all polygonal chains. Individual trajectories of separate polygonal chains do not depend on these two points, as they are the common ends of all polygonal chains. There remain the "inner" vertices of polygonal chains, namely $n-2$ points: $A_2, A_3, A_4, \ldots, A_{n-1}$. It appears that the exact polygonal chain is defined by two of these points. In addition, any two of them inevitably define some polygonal chain connecting A_1 and A_n that visits all the inner vertices A_i $(i = 2, 3, \ldots, n-1)$ once and has one self-intersection inside the circle.

Which two of the inner vertices of a polygonal chain uniquely define it? We can be assisted by the simplest cases, where the amount of base points is the smallest. The minimum possible amount of base points is 4. In this case, there is one pair of inner vertices of a polygonal chain and one polygonal chain with one self-intersection. The nature of the correspondence between them is clearly illustrated by Fig. 6.11.

The points A_2 and A_3 are the opening points of two line segments of the polygonal chain which intersect inside the circle. Nothing else can be stated about the position of the points A_2 and A_3 on the polygonal chain.

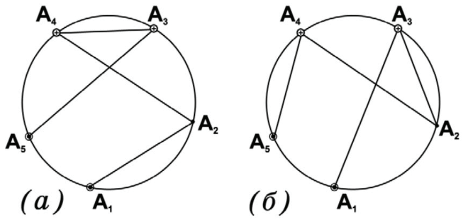

Figure 6.12. Bijection between polygonal chains and subsets. (a),(b)

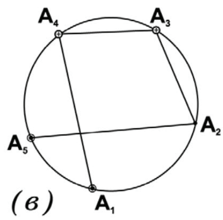

Figure 6.13. Bijection between polygonal chains and subsets. (c)

Hopefully, the case of five base points A_1, A_2, A_3, A_4 and A_5 provides more information. This time, there are three polygonal chains (C^2_{5-2}) which connect the points A_1 and A_5, have vertices in the points A_2, A_3, A_4 and one self-intersection. Here are these polygonal chains: $A_1A_2A_4A_3A_5$, $A_1A_3A_2A_4A_5$ and $A_1A_4A_3A_2A_5$ (see Fig. 6.12 and Fig. 6.13).

The set of points $\{A_2, A_3, A_4\}$ has 3 two-element subsets as well: $\{A_2, A_3\}$, $\{A_2, A_4\}$ and $\{A_3, A_4\}$. What is the unified rule which establishes a bijection between the polygonal chains and the subsets? Which pairs of inner points should be selected on each of three polygonal chains shown in Fig. 6.12, Fig. 6.13, so that provided with such pair and the ending points A_1 and A_5 one could move along the exact polygonal chain and not the other one? The rule is the following: if we move along the polygonal chain from its end A_1, then we need to select the vertex which we visit first after following along the line segment (chord) that is crossed by another chord; the second defining point is the vertex that is in a similar position concerning the other end of the polygonal chain, which is the point A_5. Hence, three pairs of vertices should be split between the polygonal chains shown in Fig. 6.12, Fig. 6.13 as follows: polygonal chain (a) – pair $(A_3; A_4)$, polygonal chain (b) – pair $(A_2; A_3)$, polygonal chain (c) – pair $(A_2; A_4)$ (clearly, the order of the components of pairs does not matter).

The above law of correspondence between the polygonal chains and the defining pairs of points remains the same for any number of base points. Below, we provide several examples of the transition from the polygonal chain to the pair of base points defining it and vice-versa for the case of nine base points.

In Fig. 6.14, the polygonal chain $A_1A_2A_3A_7A_6A_5A_4A_8A_9$ is drawn. The defining (encoding) pair of its vertices is $(A_4; A_7)$. This is explicitly provided by the law of correspondence. Moving from the point A_1 and passing the point of intersection, we get to the vertex

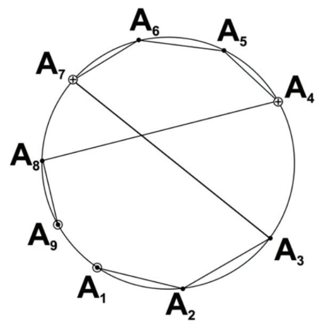

Figure 6.14. Examples of transition from the polygonal chain to the pair of base points.

A_7. Alternatively, moving from the point A_9, we get to the vertex A_4 right after the point of intersection. This is the rule of construction of the code of the polygonal chain, namely, the pair $(A_4; A_7)$.

How can we recover a polygonal chain after the defining pair of vertices (the code)? Assume that there are points $A_1, A_2, \ldots, A_9$ on a circle, and two of them A_4 and A_7 are highlighted as defining ones. How to recover the polygonal chain? The algorithm is straightforward: moving around the circle from the point A_1 counterclockwise, and from the point A_9 clockwise, connect every two adjacent base points (in our example: A_1 with A_2, A_2 with A_3, A_9 with A_8) with the chords (line segments of future polygonal chain). Moving in both directions, we stop one step away from the defining (code) vertex. In our case, we stop in the vertices A_3 and A_8. After that, we connect the vertices in which we have stopped (A_3 and A_8) with the code vertices (A_7 and A_4). Then we connect the code vertices with the polygonal chain, the vertices of which are consecutive base points ($A_4A_5A_6A_7$).

The procedure of decoding can be performed by the segmentwise construction of the corresponding polygonal chain, moving in only one direction from A_1 to A_n. So, let $A_1, A_2, \ldots, A_n$ be base points on a circle, located in this exact order if one moves in positive direction (counterclockwise). Suppose that the code of a polygonal chain with one self-intersection and ends in the points A_1 and A_n consists of the vertices A_i and A_j $(1 < i < j < n)$. The polygonal chain is constructed as follows. Draw the chord from the point A_1 to the adjacent point A_2, then to the point A_3 adjacent to it, and so on, until we get to the point A_{i-1} (one step away from the code point A_i). We connect the point A_{i-1} with A_j, and then, moving around the circle clockwise, draw the "shortest" chords A_jA_{j-1}, $A_{j-1}A_{j-2}$, and so on, until we get to the code vertex A_i. Draw the chord to A_{j+1} from it, and then again the "shortest" chords $A_{j+1}A_{j+2}$, then $A_{j+2}A_{j+3}$, and so on until the point A_n. As a result, we get the polygonal chain $A_1A_2 \ldots A_{i-1}A_jA_{j-1} \ldots A_iA_{j+1}A_{j+2} \ldots A_n$. This is the polygonal chain that correspond to the defining vertices A_i and A_j.

The established bijection between the polygonal chains in question and two-element subsets of the set $\{A_2, A_3, \ldots, A_{n-2}, A_{n-1}\}$ is the best explanation for the fact that the amount of the former is C_{n-2}^2.

It seems like a good moment to put an end to the consideration of the problem. But by doing so we will omit another instructive solution to this problem. We have already mentioned the undeniable usefulness of solving problems in different ways. Another solu-

tion is a glance at the problem from a different angle, discovery of unnoticed details and nuances. The habit of working on the problems that has already been solved is really commendable. Brilliant discoveries and pleasant surprises can be the reward for enthusiasts searching different solutions.

Having said that, we start from the very beginning once again. Let there be a circle with n "base" points $A_1, A_2, A_3, \ldots, A_n$ on it. We are required to construct a polygonal chain leading from A_1 to A_n which visits all base points once and have one self-intersection. It has the base points $A_2, A_3, \ldots, A_{n-1}$ as its only vertices.

It is easy to construct or imagine such polygonal chain. In fact, the problem is not about constructing of one or several of such polygonal chains but in answering the question: how many such polygonal chains <u>can be</u> constructed <u>in total</u>?

We have not solved this problem in this chapter in such a general setting. We considered one special case, which proved to be rather informative. We assumed that the ends of polygonal chains A_1 and A_n are located next to each other, which meant that there were no other base points on one of the arcs in which these points split the circle. In that case, it was absolutely acceptable from the context of the essence of the problem to assume that the base points lay on the circle in the order of their indices. And we proceeded with this assumption. In order to avoid any ambiguity, we additionally assume that the indices of the points increase while moving around the circle in the positive direction (counterclockwise).

Imagine that we depart from the point A_1 and plan to move along one of the polygonal chains in question. Obviously, it will not be possible for us to move along the "shortest" chords, that is, to visit the points A_i in the order in which they are placed on the circle. Indeed, if we act like that, then our trajectory will have no self-intersections. Hence, sooner or later (right from the start of after several "short" steps), we will have to "pave the way" along the "long" chord from the vertex A_k to the vertex A_{k+p}, where the number p is greater than 1 $(p > 1)$. It is reasonable to take a closer look at this "long" jump and analyze its consequences, bearing in mind that a polygonal chain should have exactly one self-intersection. The first long jump A_kA_{k+p} after the departure from the point A_1 is decisive in the sense that all further steps are predetermined. After the first long jump is made, there is no freedom of choice of the order of next destinations. We are forced to move as it is shown in Fig. 6.15. We have outlined this fact above, although we have exploited it in a different way.

And now we are going to proceed as follows. We emphasize that every polygonal chain is defined by the "nearest" to A_1 "long" line segment. Therefore, the number of polygonal chains is the same as the number of such line segments. We have established a bijection between the "long" chords (those that do not connect adjacent vertices) which can be such segments, and the polygonal chains with one self-intersection. Thus, it remains to determine which "long" chords can serve as defining segments of polygonal chains and fin their amount. Every "long" chord fits for the role of defining segment, except for those bounded by the point A_n from one side. It is straightforward to count them using, for example, the following reasoning. Two points on the circle of $n-1$ available (excluding A_n) can be connected with a chord (long or short). Hence, there are C_{n-1}^2 chords in total. There are $n-2$ short chords among them. The rest of them are long, which are in bijective

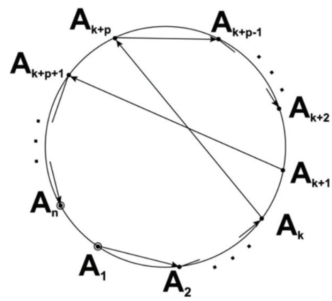

Figure 6.15. Bijection between the "long" chords and polygonal chains with one self-intersection.

correspondence with the polygonal chains. Thus, there are

$$C_{n-1}^{2}-(n-2)=\frac{(n-2)(n-3)}{2}$$

polygonal chains overall.

Problems

Problem 6.1. *There are n points on a circle. Two of them are A and B. For our convenience, we will call all these points the base points.*

How many polygonal chains satisfy all the following conditions:
a) A and B are the endpoints of a polygonal chain;
b) other base points are its vertices;
c) a polygonal chain visits all the base points once;
d) there are no other vertices.
Answer. $(n-2)!$.

Problem 6.2. *There are n base points on a circle. How many n-segment closed polygonal chains with vertices in the base points are there?*
Answer. $\frac{1}{2}\cdot(n-1)!$.

Problem 6.3. *There are n base points on a circle.*

1. *How many n-segment closed polygonal chains with vertices in the base points and no self-intersections are there?*
2. *How many n-segment closed polygonal chains with vertices in the base points and one self-intersection are there?*

Answer.

1. *1.*
2. $\frac{n(n-3)}{2}$.

Problem 6.4. *Let there be n (base) points on a circle, two of which A and B are the neighbors of a point C. How many $(n-1)$-segment polygonal chains connecting the points A and B have vertices in all other base points and one point of self-intersection which lays on the segment adjacent to the point C?*

Answer. $\frac{(n-1)(n-4)}{2}$.

Problem 6.5. *There are $n+3$ base points on a circle, two of which A and B are the neighbors of a point C. How many $(n+2)$-segment polygonal chains connecting the points A and B have vertices in all other base points and one self-intersection?*

Answer. $2C^3_{n+1}$.

Possible Scheme of Solution. Let us agree upon the notation of the base points. Let the points A, C, B follow each other clockwise. Denote other base points by $D_1, D_2, ..., D_n$ in such a way that their indices increase when moving from the point A to the point B along the arc in the positive direction (counterclockwise). Obviously, the number of polygonal chains depends on the amount of the points D_i, and thus, on n. Therefore, it is appropriate to denote this number by $\varphi(n)$. It should be noted that the introduced notation is valid only for this problem, and it has nothing to do with the Euler function, for example. Let us determine the function $\varphi(n)$.

For our convenience, we assume that the point A is the initial point of polygonal chains of interest and B is their closing point. Split all polygonal chains into three classes: those that begin with the line segment AD_1; those that begin with the line segment AC, and those that begin with any other line segment, which could be any of the chords AD_i, $i = 2, 3, 4, ..., n$.

First, consider the polygonal chains that belong to the first class, that is the chains that begin with the line segment AD_1. How many such polygonal chains exist? Obviously, $\varphi(n-1)$. Indeed, if we remove the first line segment AD_1 and the initial point A, then we get a polygonal chain with $n+1$ line segments, which satisfies all the conditions of the reduced problem that is obtained by replacing the point A for the point D_1 and removing the former point from the base points. Conversely, if we do not take into account the point A (as if it is absent) and construct various polygonal chains in question taking the point D_1 as the initial point and then attach the line segment AD_1 in front of them, then we will get all the polygonal chains from the first class. Thus, there are $\varphi(n-1)$ of them.

Now, we have to determine the number of polygonal chains in the second class. Recall that these are the chains that begin with the line segment AC. After we have reached the point C, we begin the construction of $(n+1)$-segment polygonal chain with one self-intersection, which, in addition, have its ending points C and B lying next to each other on the circle. We have found the amount of such polygonal chain in Problem 5 in the theoretical part of this Chapter. For the given number of vertices $(n+2)$, there are C^2_n such polygonal chains. Therefore, this is the amount of polygonal chains in question in the second class.

There remains the third class. In this class, there are all those paths that begin with the "long" chord AD_i ($i = 2$, or 3, or $4, \ldots$, or n). Let us observe those of them which begin with the fixed-line segment AD_k (k is a natural number from the interval $[2, n]$). As there are base points on both sides of the chord AD_k, the polygonal chain will inevitably cross it. Therefore, it should turn in the direction of the arc AD_k that contains the points D_i ($i = 1, 2, ..., k-1$) and pass these points in the order of decrease of their indices. Otherwise,

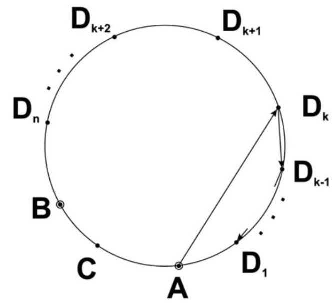

Figure 6.16. Trajectory of chain up to the point of self-intersection.

there will be at least one additional self-intersection. Hence, the beginning of the polygonal chain is:

$$AD_kD_{k-1}D_{k-2}...D_2D_1...$$

We emphasize that the defining point in this part of the polygonal chain is the point D_k. Choosing this point as our first destination when we depart from the point A, we completely define the whole trajectory of the chain up to the point of self-intersection (the segment containing it; see Fig. 6.16).

The line segment that should cross AD_k goes from D_1 to ... And its destination is the question that we have to answer. This segment can not end in any point other than C or D_{k+1}. In the former case, the chain extends passing the vertices $D_{k+1}, D_{k+2}, ..., D_n$ in this very order. In the latter case, the path continues by visiting the same vertices $D_{k+1}, D_{k+2}, ..., D_n$ in the same order, as otherwise there will appear an additional point of self-intersection (at least one). The point C, in turn, can be visited after each of the points $D_{k+1}, D_{k+2}, ..., D_n$.

As we can see, after the vertex D_1, the path to the point B might evolve following different scenarios. Is there a chance to select one vertex which unambiguously defines a path of a polygonal chain from D_1 to B? Yes. Let us agree upon the following. If there is a line segment that leads from the point D_1 to the point C, then we assume that the point C is the defining one for the polygonal chain on the whole trajectory from D_1 to B. We have already ascertained that this point (if there exists the line segment D_1C) really defines the whole further trajectory of a chain. Alternatively, if a line segment connects the points D_1 and D_{k+1}, then the defining point of the part of a chain from D_1 to B is the point D_i $(i \geq k+1)$ after which the chain first visits the point C. Knowing the point D_i, we unambiguously recover the whole path of a chain from the vertex D_1 to its ending point B. Below, we present the corresponding sequence of vertices:

$$D_1D_{k+1}...D_{i-1}D_iCD_{i+1}...D_nB.$$

Needless to say, if $i = n$, then the closing point B immediately follows C. The path of the polygonal chain from D_1 to B is shown in Fig. 6.17). Its behavior is defined by a single point: either C (Fig. 6.17) or D_i (Fig. 6.18).

Combining the above findings concerning the part of the chain from A to D_1 with the features of its ending part from D_1 to B, concentrate on two facts: the initial part of the

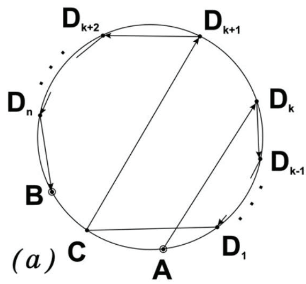

Figure 6.17. Path of the polygonal chain from D_1 to B defined by point C.

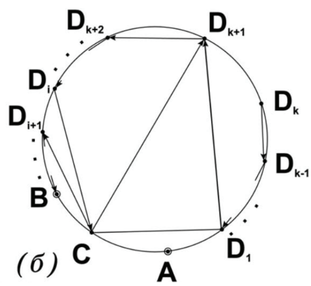

Figure 6.18. Path of the polygonal chain from D_1 to B defined by point D_i.

chain is completely defined by the point D_k $(k = 2, 3, ..., n)$; and the ending part is defined by one of the points $D_i \quad (i > k)$ or C. We conclude that the entire polygonal chain is defined by two points from the set $\{D_2, D_3, ..., D_n, C\}$. There is a bijection between the two-element subsets of this set and those polygonal chains that begin with the segment $AD_k\,(k = 2, 3, ..., n)$. Hence, there exist C_n^2 such polygonal chains.

Let us summarize the above. We managed to count the amount of polygonal chains in each of three classes which we split the entire set of chains of interest into. The fact that the number of chains in the first class is expressed by the symbol the numeric value of which is still unknown is not a big problem. Indeed, we can use it to construct the recursive formula for $\varphi(n)$, and our experience suggests that this is a valuable achievement. Here is this formula:

$$\varphi(n) = \varphi(n-1) + 2C_n^2.$$

The initial condition can be established with ease:

$$\varphi(1) = 0.$$

This can be checked directly.

We have already encountered recursive formulas of this type, and we know how to deal with them. To get the direct formula for $\varphi(n)$, one can apply the method of descent. Let us list the “copies” of our recursive formula in a column for all natural values of numeric

values of the argument from n to 2. We get the following equalities:

$$\begin{aligned}
&\varphi(n) = \varphi(n-1) + 2C_n^2, \\
&\varphi(n-1) = \varphi(n-2) + 2C_{n-1}^2, \\
&\varphi(n-2) = \varphi(n-3) + 2C_{n-2}^2, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\varphi(3) = \varphi(2) + 2C_3^2, \\
&\varphi(2) = \varphi(1) + 2C_2^2.
\end{aligned}$$

Summing them up term-wise, we get the equality

$$\varphi(n) = 2(C_n^2 + C_{n-1}^2 + C_{n-2}^2 + \dots + C_3^2 + C_2^2).$$

Thus, we have derived the direct formula for $\varphi(n)$. It can be presented in better form by the reduction of the sum of binomial coefficients. Prove that this sum equals to $2C_{n+1}^3$.

Problem 6.6. *(Extension of the previous problem) In the previous problem, we considered polygonal chains with ends A and B and one self-intersection. The vertices of these polygonal chains are located on a circle as follows: one vertex is on one hand of the chord AB and n vertices are on the other. The next step that complicates the problem is absolutely natural: instead of one base point C lying between the points A and B, we take two. We present the complete statement of the new problem below.*

There are $n+4$ base points on a circle. Two of them, A and B are the special ones: A is the starting point of polygonal chains considered below and B is their ending point. If we walk from A to B along the chord of the circle clockwise, then we will visit only two base points on our way (except for B), namely: first, we arrive at the point P, and then the point Q. Alternatively, moving from A to B along the other chord, we will come across the base points $D_1, D_2, D_3, \dots, D_n$ one by one. It is quite easy to imagine (and draw) an $(n+3)$-segment polygonal chain that connects the points A and B, has vertices in other $n+2$ base points and has one self-intersection. Much more complicated is to find the answer to the question: how many such polygonal chains exist?

The problem is about finding the above amount.

In other words, we need to find out how many ways are there to construct a the trajectory from A to B along with the chords of the circle that connect the base points with each other, ensuring that the following conditions are fulfilled:

1. *the path must visit every base point once;*
2. *the trajectory should have one self-intersection.*

Answer. $3C_{n+2}^4 + n + 1$.

Sketch of Solution. Let us split all sought polygonal chains into 4 groups: chains that begin with the segment AD_1; chains that begin with the segment AP; chains that begin with the segment AQ; finally, chains that begin with the segments AD_k for $k > 1$.

Let $\psi(n)$ be the wanted number. We will determine the number of polygonal chains in each class.

Class I. The polygonal chains that begin with the segment AD_1 are in bijective correspondence with the polygonal chains that connect the points D_1 and B (provided that the

point A and the segment AD_1 are removed) and possess all the properties required in the problem. If we remove the point A and delegate its duties to the point D_1, then we find ourselves in the original setting except for the fact that the amount of interim base points D_i is less by one. Therefore, there will be $\psi(n-1)$ polygonal chains. And this is the amount of polygonal chains on the first class.

Class II. The polygonal chains of this class begin with the segment AP. If this segment and the point A are detached from them, then what we get are essentially the polygonal chains considered in the previous problem. As we know, there are $2C_{n+1}^3$ of them.

Class III consists of only one polygonal chain

$$AQPD_1D_2...D_nB.$$

Class IV. The polygonal chains of this class are in a bijective correspondence with the triplets of vertices from the set $\{P, Q, D_2, D_3, ..., D_n\}$. Let us clarify this correspondence with an example. Let $n = 9$. The triplet $\{D_3, D_5, D_8\}$ has the polygonal chain

$$AD_3D_0D_1D_4D_5PD_6D_7D_8QD_9B$$

corresponding to it. The law of correspondence is as follows. If there are no letters P and Q in a triplet, then the letter D with the smallest index (of the triplet) is the end of the first segment of the polygonal chain, after the letter D with the second smallest index there should follow the letter P, and the letter D with the greatest index should be followed by the letter Q. The positions of other letters (vertices of polygonal chain) are unambiguously defined by the properties of polygonal chains of interest: one self-intersection, $n+3$ line segments, B is the ending vertex.

If a triplet consists of the letter P and two letters D (e.g., D_k and D_i, $k < i$), then D_k is the endpoint of the polygonal chain, P is the vertex following D_1, and Q is the vertex following D_i. For example, for $n = 9$ the triplet $\{D_4, D_7, P\}$ is the code of the polygonal chain

$$AD_4D_3D_2D_1PD_5D_6D_7QD_8D_9B.$$

If a triplet contains Q, then the polygonal chain has the segment PQ.

For example, for $n = 9$, the triplet $\{D_4; D_6; Q\}$ defines the polygonal chain

$$AD_4D_3D_2D_1D_5D_6PQD_7D_8D_9B,$$

and the triplet $\{D_4, P, Q\}$ has the chain

$$AD_4D_3D_2D_1PQD_5D_6D_7D_9B$$

corresponding to it.

It appears that the fourth class contains

$$C_{n+1}^3$$

polygonal chains.

In total, there are

$$\psi(n) = \psi(n-1) + 3C_{n+1}^3 + 1$$

polygonal chains

Having drawn the figure, we determine that $\psi(1) = 2$.

We proceed in a familiar manner. Replicate the above recurrence relation for all values from n to 2:

$$\begin{aligned}
&\psi(n) = \psi(n-1) + 3C_{n+1}^3 + 1,\\
&\psi(n-1) = \psi(n-2) + 3C_n^3 + 1,\\
&\psi(n-2) = \psi(n-3) + 3C_{n-1}^3 + 1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\psi(4) = \psi(3) + 3C_5^3 + 1,\\
&\psi(3) = \psi(2) + 3C_4^3 + 1,\\
&\psi(2) = \psi(1) + 3C_3^3 + 1.
\end{aligned}$$

Summing them up term-wise, taking into account that $\psi(1) = 2$ and reducing the sum

$$C_3^3 + C_4^3 + C_5^3 + \dots + C_n^3 + C_{n+1}^3,$$

we get the answer:

$$\psi(n) = 3C_{n+2}^4 + n + 1.$$

The abovementioned sum can be reduced as follows:

$$\begin{aligned}
&C_3^3 + C_4^3 = C_4^4 + C_4^3 = C_5^4;\\
&C_5^4 + C_5^3 = C_6^4;\ C_6^4 + C_6^3 = C_7^4;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&C_n^4 + C_n^3 = C_{n+1}^4;\ C_{n+1}^4 + C_{n+1}^3 = C_{n+2}^4.
\end{aligned}$$

Problem 6.7. *(Generalization of several previous problems).*

The last problem thoroughly considered in the theoretical part of this The chapter was one about polygonal chains with one self-intersection and fixed vertices lying on a circle. In addition, the endpoints of polygonal chains were assumed to be adjacent base points. In two previous problems (5 and 6) we considered similar combinatorial situations, namely, the polygonal chains with one self-intersection the endpoints of which were separated by one or two vertices (the other arc could contain unlimited but fixed amount of base points). Now, we suggest applying the obtained experience to solve the problem about the polygonal chains with fixed endpoints and vertices and one self-intersection in the most general setting.

Consider a circle where there are points A and B, m other base points $C_1, C_2, \dots, C_m$ on one of the arcs which A and B split the circle into, and n base points on the other arc. Denote the latter points by the letters $D_1, D_2, \dots, D_n$. In order to develop a clear image of the situation, assume that the points $C_1, C_2, \dots, C_m$ are located one by one clockwise on the way from A to B along the arc of the circle, and the points $D_1, D_2, \dots, D_n$ are located similarly on the other side arc AB in the increasing order when moving counterclockwise.

The task is to find the amount of different $(m+n+1)$-segment polygonal chains with one self-intersection, the endpoints of which are the points A and B, and the vertices are other $m+n$ base points $C_1, C_2, \dots, C_m$, $D_1, D_2, \dots, D_n$.

Answer. $(m+1)C_{m+n}^{m+2} + (n+1)C_{m+n}^{n+2}$.

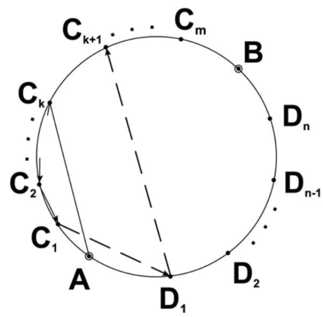

Figure 6.19. AC_k is the initial segment of a chain.

Solution. Assume that the point A is the starting point of polygonal chains and B is their ending point. We will need this assumption later on when we will deduce the recurrence relation for the sought number. In turn, we will denote this number by $\lambda(m;n)$, emphasizing that it depends on two natural (or zero) parameters m and n, which denote the amounts of base points on two arcs bounded by the points A and B.

What are we required to do? Ideally, we have to find (in any suitable way) the direct formula for $\lambda(m;n)$. This is our ultimate plan, or more exactly, the ultimate goal as we have no plan at the moment. In the worst case, we can confine to the recursive formula for the sought value, although the applicability of such formula should be thoroughly inspected because of the dependence of $\lambda(m;n)$ on two parameters instead of one. This can be achieved by an appropriate definition of the initial conditions.

The special cases of the current problem that we have considered before advise that we attempt to find the recurrence relation first. To this end, as in the previous problem, we split all polygonal chains in question into 4 classes.

The first class will contain the polygonal chins beginning with the segment AC_1. How many such polygonal chains are there? Clearly, there are $\lambda(m-1;n)$. Indeed, such polygonal chains are in a "natural" bijective correspondence with those polygonal chains which connect C_1 and B, have vertices in other base points C_i $(i = 2, \ldots, m)$ and D_j $(j = 1, 2 \ldots, n)$ and have one self-intersection. The family of original polygonal chains of the first class turns into the family of polygonal chains with one self-intersection which connect the points C_1 and B, when the segment AC_1 and the vertex A are removed from each chain of the former family.

The second class is composed of all those chains that begin with the segment AD_1. There are $\lambda(m-1;n)$ of them. In order to justify this conclusion, one needs to modify the reasoning in the previous paragraph by replacing m for n and the letters $C_1, C_2, \ldots, C_m$ for $D_1, D_2, \ldots, D_n$ and vice versa.

The third class includes all polygonal chains which begin with any segment AC_k, except for AC_1. The amount of such polygonal chains can be given by a direct formula. Let us deduce it.

Let AC_k be the initial segment of a chain, where k is a fixed natural number greater than 1 (Fig. 6.19). As there are base points on both sides of the line AC_k, the polygonal chain will definitely cross the chord AC_k. According to the statement of the problem, the chain

should have one self-intersection. This means that after the point C_k it should necessarily "visit" the points C_{k-1}, C_{k-2}, ..., C_1 in this exact order, that is, C_{k-1} comes first, then C_{k-2} and so on, up to C_1. In other words, if a chain begins with the segment AC_k, then it should evolve as follows:

$$AC_kC_{k-1}C_{k-2}\dots C_2C_1\dots$$

After the vertex C_1, there is no uniqueness in the path. It can behave in various ways, although not randomly. There are two restrictions limiting further options:

first, it should visit all remaining base points;

second, after crossing the chord AC_k (which happens immediately after the chain passes the vertex C_1) it should not cross itself anymore.

So, how can and how can not the chain behave after reaching the pointC_1?

There are $m-k$ points C_i (from C_{k+1} to C_m) and all n points D_j left for it to visit in order to reach the point B (and stop at it). The points C_i $(i = k+1, k+2, \dots, m)$ should be passed in the order of increase of their indices. Otherwise, another self-intersection is inevitable. The same concerns to the points D_j $(j = 1, 2, 3, \dots, n)$. As to the relative order of the points C_i and D_j on the chain's path from C_1 to B, many options are possible. And this variability provides that there are many suitable chains instead of one. How many such chains exist? This question is rather easy. The amount of polygonal chains in every such family is defined by the number of ways to line up $m-k$ letters C and n letters D. We already know that there are C^n_{m+n-k} ways to do this. Thus, we are very close to the answer to the question about the number of polygonal chains in the third class now. The index k can gain any values from 2, 3, ..., m. Therefore, there are

$$C^n_{m+n-2} + C^n_{m+2n} + \dots + C^n_{m+n-(m-1)} + C^n_{m+n-m}$$

polygonal chains in the third class. Reducing the above sum (we have reduced the sums of this type before), we get the final answer: there are

$$C^{n+1}_{m+n-1}$$

chains in the third class.

It is worth noting that it is possible to count the chains of the third class in such a way that directly provides the answer in the brief form C^{n+1}_{m+n-1}. There are $m+n$ vertices in any polygonal chain altogether. Naming them in a certain order, we get one or another polygonal chain. As it comes from the above research, any chain of the third class is uniquely defined by the positions of the following vertices: $C_1, D_1, D_2, \dots, D_n$. In addition, the number of the vertex should be greater than 1 and less than the one of the vertex D_1 (recall that the latter is the least of the numbers of the vertices D_j). Thus, if we choose any $(n+1)$-element subset of the set $\{2, 3, , \dots, m+n\}$, then it will define a certain chain. Conversely, any chain of the third class has an $(n+1)$-element subset of the above set corresponding to it. It appears that there is a bijection between the polygonal chains of the third class and $(n+1)$-element subsets of the set $\{2, 3, \dots, m+n\}$. Hence, there are C^{n+1}_{m+n-1} chains of this type in total.

Remark. If the first segment of a chain is AC_k, then the vertex C_1 is in the k-th position (among the inner vertices). Thus, the number of its position in the sequence of inner vertices of a chain (excluding the point A) can not exceed m. We did not make such a remark in

the considerations of the previous paragraph. Prove that there was no need for it as this condition is fulfilled automatically.

As to the chains of the fourth class, it is straightforward to count them. It appears that in the statement of the problem, the roles of the numbers m and n are symmetrical. Similarly to the roles of the sequences of the base points $C_1, C_2, \ldots, C_m$ and $D_1, D_2, \ldots, D_n$. The fourth class includes all those polygonal chains which begin with any segment AD_j, except for AD_1. Clearly, we can determine their number with ease. It suffices to replace the symbol m for n in the symbol that denotes the number of chains in the third class, and vice versa. Thus, the first class contains

$$C_{m+n-1}^{m+1}$$

polygonal chains. It is a right time to summarize the obtained results. The mini problem is now solved. We have deduced the recurrence relation for the sought amount $\lambda(m;n)$:

$$\lambda(m;n) = \lambda(m-1;n) + \lambda(m;n-1) + C_{m+n-1}^{m+1}. \tag{6.12}$$

Now, we have to investigate how this formula works. We already know that an appropriate basis is needed for a recurrence relation to producing certain numbers one by one. This basis is the initial conditions. Our formula deals with two parameters: m and n. Both can gain zero and natural values: $0, 1, 2, 3, \ldots$. Which initial conditions does this formula require? The answer becomes obvious, when we place the values of $\lambda(m;n)$ in their "natural" order, which is in the form of a two-dimensional array infinitely extending to the bottom and the right:

Table 6.2. Table for $\lambda(m;n)$

$\lambda(0;0)$	$\lambda(0;1)$	$\lambda(0;2)$	$\lambda(0;3)$	$\lambda(0;4)$	$\ldots$
$\lambda(1;0)$	$\lambda(1;1)$	$\lambda(1;2)$	$\lambda(1;3)$	$\lambda(1;4)$	$\ldots$
$\lambda(2;0)$	$\lambda(2;1)$	$\lambda(2;2)$	$\lambda(2;3)$	$\lambda(2;4)$	$\ldots$
$\lambda(3;0)$	$\lambda(3;1)$	$\lambda(3;2)$	$\lambda(3;3)$	$\lambda(3;4)$	$\ldots$
$\lambda(4;0)$	$\lambda(4;1)$	$\lambda(4;2)$	$\lambda(4;3)$	$\lambda(4;4)$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Taking into account the recursive formula for $\lambda(m;n)$, we realize that every number in the table 6.2, except for those standing in the first column and the first row, is expressed in a certain way (in the way that is prescribed by the formula) with its left and top neighbors. For example,

$$\lambda(3;2) = \lambda(2;2) + \lambda(3;1) + C_4^3 + C_4^4.$$

The recurrence relation will become operative immediately after we define all the values on the bounds (top row and left column) of the table, namely the values of $\lambda(0;n)$ and $\lambda(m;0)$. We know all of them. Indeed, $\lambda(0;n)$ is nothing else but $\alpha(n+2)$ from Problem 5 of the theoretical part of this chapter. Hence,

$$\lambda(0;m) = C_n^2.$$

As to the numbers $\lambda(m;0)$, obviously, they are equal to the corresponding numbers $\lambda(0;n)$. The word "corresponding" here means that they have the same value of the parameter. In other words,

$$\lambda(m;0) = \lambda(0;m).$$

Why? Because both numbers have the same combinatorial meaning. Both are the solutions to the same problem, namely: how many $(m+1)$-segment polygonal chains with endpoints A and B and one self-intersection are there, if all other base points (which are the vertices of the chains) lay on one of the arcs AB.

Therefore, the basis for our recurrence relation (R) are the following initial conditions:

$$\lambda(m;0) = \lambda(0;m) = C_m^2 \ \ (m = 0, 1, 2, ...,).$$

Basing on them, we are able (at least potentially) to compute any number $\lambda(m;n)$. It should be emphasized that for any values of the parameters m and n, the equality

$$\lambda(m;n) = \lambda(n;m),$$

holds, which can be explained either from combinatorial or formally arithmetical point of view. Combinatorically, both numbers $\lambda(m;n)$ and $\lambda(n;m)$ have the same meaning: they denote the amount of $(m+n+1)$-segment polygonal chains which connect the points A and B on the circle, have m vertices on one arc AB of the circle, n vertices on the other arc and have one self-intersection. Arithmetical explanation is straightforward as well. It is the corollary of the symmetry w.r.t. m and n of the recursive formula and the initial conditions.

Is it possible to deduce the direct formula for the numbers $\lambda(m;n)$, that is, the formula which expresses the value of $\lambda(m;n)$ directly with the values of the parameters m and n? Let us attempt to derive it.

In two previous problems we managed to find the formula for $\lambda(1;n)$ and $\lambda(2;n)$. Here are these formulas:

$$\lambda(1;n) = 2C_{n+1}^3, \ \ \lambda(2;n) = 3C_{n+2}^4 + (n+1).$$

In addition, we know the initial conditions, which include the direct formulas for $\lambda(0;n)$:

$$\lambda(0;n) = C_n^2 \ \ (n = 0, 1, 2, 3, \ldots).$$

Let us take a careful look at the table of values of $\lambda(0;n)$, $\lambda(1;n)$ and $\lambda(2;n)$:

Table 6.3. Table for $\lambda(m;n), m = 0,1,2$

$\lambda(0;n)$	$\lambda(1;n)$	$\lambda(2;n)$
$1 \cdot C_n^2$	$2C_{n+1}^3$	$3C_{n+2}^4 + (n+1)$

Basing on the tables 6.2 and 6.3, is it possible to guess the formula for $\lambda(m;n)$?
Concerning the numbers

$$1 \cdot C_n^2, \ 2 \cdot C_{n+1}^3, \ 3 \cdot C_{n+2}^4, \tag{6.13}$$

their connection with the value of the parameter m is explicitly visible and it suggests that the above sequence extends as follows:

$$4C_{n+3}^{5},\ 5C_{n+4}^{6},\ \ldots,\ (m+1)C_{n+m}^{m+2},\ \ldots \tag{6.14}$$

However, the euphoria evaporates once the summand $n+1$ arises in the formula $\lambda(2;n)$. Why is there no such summand for the smaller values of m, while it appears for $m=2$? How will it change for greater values of m? And can it happen that other additional summands arise at some stage of growth of m? These are the most important questions, arising immediately as we see the new summand. A thorough investigation of these questions will, undoubtedly, be helpful for the ideas of new hypotheses about the structure of the direct formula for $\lambda(m;n)$. It can not be

$$(m+1)\cdot C_{m+n}^{m+2}.$$

Why? Because this formula is not symmetrical w.r.t. m and n. Therefore, even before the second summand arose in the formula for $\lambda(2;n)$, its emergence for higher values of m could have been predicted. If we forecast the extension of the sequence (6.13), (6.14) correctly, then there are grounds to expect that the sought direct formula $\lambda(m;n)$ contains two summands, and one of them symmetrically balances the other. If our predictions are correct, then the direct formula for $\lambda(m;n)$ should be as follows:

$$\lambda(m;n) = (m+1)\cdot C_{m+n}^{m+2} + (n+1)\cdot C_{m+n}^{n+2}. \tag{6.15}$$

Thus, we have formulated the hypothesis. Its fate is now in the hands of the "general" recursive formula and the initial conditions.

Let us verify if our hypothetical formula is coherent with the initial conditions. Let us recall them:

$$\lambda(0;n) = C_n^2,\ \lambda(m;0) = C_m^2.$$

Below, is the result of the application of the major "candidate" for the direct formula:

$\lambda(0;n) = 1\cdot C_n^2 + (n+1)\cdot C_n^{n+2} = C_n^2$, because $C_n^{n+2} = 0$;

$\lambda(m;0) = (m+1)C_m^{m+2} + 1\cdot C_m^2 = C_m^2$, because $C_m^{m+2} = 0$.

Hence, the hypothetical formula survives the test for initial conditions.

Now, it comes the turn of the ultimate and decisive test, which is the examination with recursive formula (6.12). We have:

$$\lambda(m-1;n) + \lambda(m;n-1) + C_{m+n-1}^{n+1} + C_{m+n-1}^{m+1} =$$

$$= m\cdot C_{m+n-1}^{m+1} + (n+1)\cdot C_{m+n-1}^{n+2} +$$

$$+(m+1)\cdot C_{m+n-1}^{m+2} + n\cdot C_{m+n-1}^{n+1} + C_{m+n-1}^{n+1} + C_{m+n-1}^{m+1} =$$

$$= (m+1)\cdot(C_{m+n-1}^{m+1} + C_{m+n-1}^{m+2}) + (n+1)\cdot(C_{m+n-1}^{n+1} + C_{m+n-1}^{n+2}) =$$

$$= (m+1)\cdot C_{m+n}^{m+2} + (n+1)C_{m+n}^{n+2} = \lambda(m;n).$$

Thus, we have derived the expected result. The formula that we have guessed obeys the law of recurrence relation (6.12). From now on we can claim: the problem is solved by the direct formula (6.15).

Problem 6.8. *1) How many ways are there to choose three numbers out of n natural numbers $\{1, 2, 3, ..., n\}$, so that the difference of two smallest of them does not exceed 2?*

2) How many ways are there to choose three numbers out of n natural numbers $\{1, 2, 3, ..., n\}$, so that the difference of two smallest of them does not exceed K_1, and the difference between two greatest does not exceed K_2 $(K_1 > K_2 \in N)$?

Answer. *1)* C^3_{n-3}*; 2)* $C^3_{n-K_1-K_2}$.

Solution. 2) Along with the set $A = \{1, 2, 3, ..., n\}$, consider the set $B = \{1, 2, 3, ..., n-K_1-K_2\}$. (The set B becomes fictional if $K_1+K_2 \geq n$. But in this case, no required triplets of numbers exist. It is obvious that such triplets only exist when $n \geq K_1+K_2+3$). Let $S_3(A)$ be the set of all those triplets of numbers which are described in the statement of the problem, and $S_3(B)$ is the set of all 3-element subsets of the set B. Below, we will ascertain that both sets contain the same amounts of elements. To this end, let us establish a bijection between these set. The law of correspondence is as follows:

match every triplet $(x; y; z) \in S_3(A)$ $(x < y < z)$ with the triplet (3-element subset) $\{x, y-K_1, z-K_1-K_2\}$ from $S_3(B)$.

Clearly, one needs to check if the triplet $\{x, y-K_1, z-K_1-K_2\}$ really belongs to B, given that the triplet $(x; y; z)$ belongs to $S_3(A)$. A triplet belongs to $S_3(A)$ if and only if its components are natural, the greatest of them does not exceed n, and the differences of two least and two greatest of them do not exceed K_1 and K_2 respectively. A triplet belongs to the set $S_3(B)$, when its components are natural and the greatest of them does not exceed $n-K_1-K_2$. So, let $(x; y; z) \in S_3(A)$. This means that $x; y; z \in N$ and $y-x > K_1$, $z-y > K_2$, $z \leq n$. Therefore, we conclude that $x, y-K_1, z-K_1-K_2 \in Z$ and

$$x < y-K_1 < z-K_1-K_2 \leq n-K_1-K_2,$$

which yields $\{x, y-K_1, z-K_1-K_2\} \in S_3(B)$.

Hence, our law really establishes a mapping of the set $S_3(A)$ in the set $S_3(B)$. This mapping is clearly an injective one (if $(x; y; z) \neq (u; v; t)$, then $\{x, y-K_1, z-K_1-K_2\} \neq \{u, v-K_1, t-K_1-K_2\}$). It is also a surjective mapping. Indeed, let $\{a, b, c\} \in S_3(B)$, that is $a, b, c \in N$ and $a < b < c \leq n-K_1-K_2$. Then $a < b+K_1 < c+K_1+K_2 \leq n$, hence, $(a; b+K_1; c+K_1+K_2) \in S_3(A)$. It is easy to see that the triplet $(a; b+K_1; c+K_1+K_2)$ turns into the triplet $\{a, b, c\}$ by the introduced law, which evidences that the correspondence is a surjection.

The constructed bijection provides that the sets $S_3(A)$ and $S_3(B)$ contain the same amounts of elements. It remains to perform a straightforward calculation: the second set is composed of all 3-element subsets of the set B, and thus contains $C^3_{n-K_1-K_2}$ elements.

Problem 6.9. *There are $n+2$ base points on a circle: $A, D_1, D_2, D_3, ..., D_n, B$. They are located in the stated order when moving around the circle counterclockwise. We have already learned that even for small values of n, many $n+1$-segment polygonal chains can be constructed which begin in the point A, "visit" every point D_k (these points are the vertices of polygonal chains) and end in the point B. We are interested in the chain with two self-intersections now. Attempt the following questions.*

1. *Which is the smallest value of n, for which there exists at least one polygonal chain with two self-intersections?*

2. *It is known that a polygonal chain has two self-intersections and both points of self-intersection lay on the first line segment. Draw all such polygonal chains for those two smallest values of n for which they exist. For an arbitrary value of n (in the general case), attempt defining the sequence of vertices of such polygonal chain when moving from its beginning (the point A) to the end (the point B). How many such chains exist in total?*

3. *Moving along the chain from A to B, we highlight the line segments which contain the points of self-intersection. It appears that the first segment contains both points of self-intersection. Write down the sequence of vertices of such polygonal chain, moving from A to B. How many such chains exist?*

4. *Moving from A to B along the chain with two self-intersections, we observe which segments do the points of self-intersection belong to. It appeared that we passed the segment with one point of self-intersection first (skipping the segments that do not contain such points), then the one with two points of self-intersection, and then the segment with one such point again. How many such polygonal chains exist? Draw 2-3 of them for the case $n = 7$.*

5. *How many chains with two self-intersections have the points of self-intersection lying one per segment on four different line segments of a chain?*

6. *How many different polygonal chains with two self-intersections are there?*

Answer.

1. $n = 3$;
2. C_{n-1}^2;
3. C_n^3;
4. C_{n-1}^3;
5. C_{n-1}^4;
6. $C_{n+1}^4 + C_n^3$.

Solution. 2) The first segment of the chain can not be "short" (its endpoints are not adjacent base points). Therefore, this segment is one of the chords AD_k, where $k > 1$. It should be crossed by two other segments. Hence, the next (second) segment is necessarily D_kD_{k+1}, and thus $k < n$. The further path of the chain is sketched in Fig. 6.20.

The sequence of its vertices is presented below:

$$AD_kD_{k+1}\ldots D_mD_{k-1}D_{k-2}\ldots D_2D_1D_{m+1}\ldots D_nB.$$

The path of the chain is uniquely defined by two of its vertices: D_k and D_m. Thus, the indices k and m are chosen from $2, 3, 4, \ldots, n$. There are C_{n-1}^2 ways to choose them. And this is the amount of different polygonal chains in question.

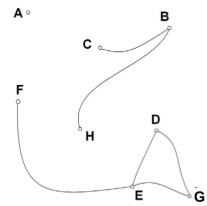

Figure 6.20. The sequence of vertices.

3) Any polygonal chain of the stated type can begin wither with the segment AD_1, or with the segment AD_k $(k > 1)$. Let $\mu(n)$ be the wanted amount of chains. There are $\mu(n-1)$ of those which begin with AD_1, and as we have just determined, C_{n-1}^2 chains that begin with AD_k $(k > 1)$. Then

$$\mu(n) = \mu(n-1) + C_{n-1}^2.$$

The initial condition can be determined directly:

$$\mu(2) = 0$$

(2 is the greatest value of n for which there are no chains of the required type).

Now, we apply the familiar "descend" approach:

$$\begin{array}{l}
\mu(n) = \mu(n-1) + C_{n-1}^2, \\
\mu(n-1) = \mu(n-2) + C_{n-2}^2, \\
\mu(n-2) = \mu(n-3) + C_{n-3}^2, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\mu(4) = \mu(3) + C_3^2, \\
\mu(3) = \mu(2) = C_2^2.
\end{array}$$

Summing up the above equalities term-wise, we get:

$$\mu(n) = C_{n-1}^2 + C_{n-2}^2 + C_{n-3}^2 + \ldots + C_3^2 + C_2^2.$$

We have experience of reduction of the sums of the above type. But still...

Express the well-known recurrence relation for the binomial coefficients

$$C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$$

in the form

$$C_{n-1}^{k-1} = C_n^k - C_{n-1}^k$$

and apply it to each summand of our sum:

$$\begin{array}{l}
C_{n-1}^2 = C_n^3 - C_{n-1}^3, \\
C_{n-2}^2 = C_{n-1}^3 - C_{n-2}^3, \\
C_{n-3}^2 = C_{n-2}^3 - C_{n-3}^3, \\
\dots\dots\dots\dots\dots\dots\dots \\
C_4^2 = C_5^3 - C_4^3, \\
C_3^2 = C_4^3 - C_3^3, \\
C_2^2 = C_3^3.
\end{array}$$

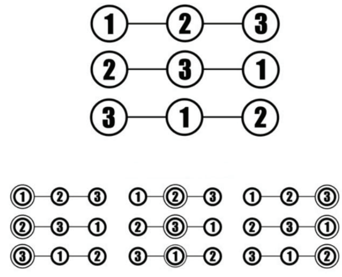

Figure 6.21. Path of the chain from A to B.

Summing up the above equalities term-wise, we arrive at

$$\mu(n) = C_n^3.$$

The answer encourages searching for another solution, just to "suit the formula". An alternative solution is presented below.

Any polygonal chain is unambiguously defined by 3 points from the set $\{A, D_1, D_2, D_3, D_{n-1}, D_n\}$, given an additional condition is satisfied: two of them with the least indices (assume A has index 0) should not be adjacent. The first two of these points are the ends of the segment that carries two points of self-intersection, and the third is the beginning of the reverse movement of a chain (in Fig. 6.20, this point is denoted by D_m). The correspondence between the considered triplets and the polygonal chains in question is bijective. Hence, there exist as many chains as there are triplets. Now, we are facing the question: how many ways are there to choose a triplet $(k; k+p; k+p+q)$, where $p > 1$, $q \geq 1$, from the set of $n+1$ numbers $\{0, 1, 2, 3, ..., n-1, n\}$? The answer is C_n^3 (see problem 7).

4) First, find the number of those polygonal chains (of the required type) which begin with a "long" segment. Let AD_k be this segment. It can contain one point of self-intersection, so the further evolution of the chain is as follows:

$$AD_kD_{k-1}...D_2D_1.$$

The segment beginning with the point D_1 will cross the segment AD_k. But it must contain one more point of self-intersection, hence, its other endpoint can not be D_{k+1}. Thus, the next segment of the chain is D_1D_{k+s}, where $s \geq 2$. The further path of the chain is predefined:

$$D_1D_{k+s}D_{k+s-1}...D_{k+1}D_{k+s+1}D_{k+s+2}...D_nB.$$

The path of the chain from A to B is shown in Fig. 6.21. It is uniquely defined by two points: D_k and D_{k+s}. In addition, we require $k > 1$ and $s > 1$. The above considerations establish a bijection between the chains and pairs of numbers $(k; k+s)$, $k > 1, s > 1$. Therefore, the question about the number of chains has been reduced to the following question: how many pairs of numbers $(k; k+s)$ can be created with the numbers $\{2, 3, ..., n\}$ if the components of a pair should differ at least by 2 ($s \geq 2$)? We have the experience of solving problems of this type (see problem 7). Although the current problem, in which only the

value 1 is restricted for s, can be solved quite easily. C_{n-1}^2 is the number of pairs of adjacent numbers. Then the wanted amount is:

$$C_{n-1}^2-(n-2)=C_{n-1}^2-C_{n-2}^1=C_{n-2}^2.$$

It is time to consider the question of the problem in its entirety.

Denote the sought value by $\tau(n)$. Any polygonal chain of interest begins either with the segment AD_1 or with a "long" segment. The amount of chain of the first type (with the initial segmentAD_1) is $\tau(n-1)$, and the number of the chains of the latter type is C_{n-2}^2, as we have just determined. Thus, we have the following recurrence relation for $\tau(n)$

$$\tau(n)=\tau(n-1)+C_{n-2}^2.$$

The initial condition

$$\tau(3)=0$$

is determined directly (3 is the greatest value of n for which $\varphi(n)=0$).

The direct formula can be obtained by, for example, the method of "descend":

$$\begin{aligned}
&\tau(n)=\tau(n-1)+C_{n-2}^2,\\
&\tau(n-1)=\tau(n-2)+C_{n-3}^2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\tau(6)=\tau(5)+C_4^2,\\
&\tau(5)=\tau(4)+C_3^2,\\
&\tau(4)=\tau(3)+C_2^2.
\end{aligned}$$

Summing the above equalities term-wise, we get

$$\tau(n)=C_{n-2}^2+C_{n-3}^2+\ldots+C_4^2+C_3^2+C_2^2=C_{n-1}^3.$$

The answer contains the binomial coefficient again. This evidence is in favor of the fact that a reasonable observation can help to derive the answer in an absolutely different way. Indeed, C_{n-1}^3 is the amount of 3-subsets of an $(n-1)$-element set. Therefore, there exists a bijection between the sought chains and 3-element subsets of an $(n-1)$-element set. If we were able to suspect such a relation in the beginning, then we could come up with the most efficient and elegant solution. We would have got the answer in the form C_{n-1}^3 straight ahead, skipping the recurrence relation and other technicalities. However, it is still tempting to investigate the nature of the relation between 3-element subsets of an $(n-1)$-element set to chains in question. Clearly, this should be connected somehow with the existence of a triplet of defining vertices in each polygonal chain. In order to verify this idea, let us find out which essential structural specifics are inherent to the polygonal chains in question. Imagine that we depart from the point A to make a journey along with one of these polygonal chains, during which we will make notes about our movement from vertex to vertex. Obviously, our notes will depend on the exact chain which we follow. However, the reports about journeys along different polygonal chains will have much in common, and these common features are the object of our further investigation.

Departing from the point A, we first pass the chain of "short" segments (each of which is bounded by adjacent base points). The report about this (opening) part of the journey can be expressed by the following chain of vertices:

$$AD_1D_2...D_l.$$

This is the first stage of our trip. Besides, this opening chain can be short or long. In particular, it can contract to one vertex A. Then the report on the opening stage of the trip is just A.

The "long" segment should follow. The chain makes a jump to the vertex D_k ($k > l+1$ if the jump occurs from the vertex D_l; if the jump is from the point A, then $k > 1$). Then the chain continues in the reverse direction with short steps:

$$D_kD_{k-1}...D_{l+1}$$

(the last vertex is D_1 if $l = 0$, that is if the long jump has been made from the point A). After this, the uniqueness of the path fails again. From the point D_{l+1}, the chain has to make a long jump again and its point of destination D_m has to be defined. There is only one restriction on it: $m > k+1$. As soon as we choose the exact value of m, the further extension of the polygonal chain proceeds up to its end with no alternative:

$$D_mD_{m-1}...D_{k+1}D_{m+1}D_{m+2}...D_nB.$$

It appears that three vertices are defining for the evolution of the chain:

D_l (A if $l = 0$) is the position of the first long jump;

D_k is its point of destination;

D_m is the endpoint of the second long jump.

They are chosen from the set

$$\{A, D_1, D_2, D_3, ..., D_n\}$$

in such a way that there is no adjacent among them. The number of such triplets of points is the same as the amount of 3-element subsets of an $(n-1)$-element set (see problem 8). That is why there are C_{n-1}^3 polygonal chains.

5) The configuration of the polygonal chain in question should be as in Fig. 6.22.

It defining vertices are: D_l, D_k, D_m, D_s. If they are specified, then the chain is completely defined. The components of a defining quartet belong to the set

$$\{A, D_1, D_2, ..., D_m\}, \tag{6.16}$$

comprising $n+1$ points. However, the amount of chains is not C_{n+1}^4, because the points of a defining quartet have two peculiarities: the first two (D_l and D_k) and the last two (D_m and D_s) of them can not be adjacent. In other words, the following inequalities should hold:

$$k > l+1, \; s > m+1.$$

To choose such a quartet of vertices from the set (6.16) is the same as to choose a quartet of numbers $< l;\ k;\ m;\ s >$ from the set

$$E = \{0, 1, 2, 3, ..., n\},$$

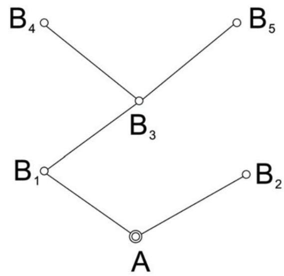

Figure 6.22. Configuration of the polygonal chain.

which (quartet) has the following properties:

$$l+1<k,\ k<m,\ m+1<s.$$

How many such quartets exist? The answer will be provided by a bijection between them and all quartets $<a;b;c;d>$ $(a<b<c<d)$ from the set

$$F=\{0,1,2,3,...,n-2\}.$$

The law of bijective correspondence is:

$$\langle l;k;m;s\rangle \leftrightarrow \langle l;k-1;m-1;s-2\rangle. \tag{6.17}$$

Let us ascertain that it indeed establishes a bijection those quartets $<l;\ k;\ m;\ s>$ of numbers from E the first and the last pairs of components of which are not adjacent numbers, and all quartets with increasing components from F.

Let $<l;\ k;\ m;\ s>$ be a quartet with the components from E and $l+1<k,\ k<m,\ m+1<s$. Then $l<k-1,\ k-1<m-1$ and $m-1<s-2$, hence, the numbers of the second quartet form an increasing sequence. In addition, $s-2\le n-2$, because $s\le n$. This yields that $<l;k-1;m-1;s-2>$ is a quartet with the components from F. Conversely, if $<a;b;c;d>$ is a quartet if numbers from F and $a<b<c<d$, then the components of the quartet $<a;b+1;c+1;d+2>$ belong to E, form an increasing sequence, and the inequalities $b+1>a+1$ and $d+2>(c+1)+1$ hold.

Hence, the rule (R) really establishes a bijection between the quartets of interest and all 4-element subsets of the set F. The conclusion is that there are C_{n-1}^4 wanted polygonal chains.

6) Hint. Summing up the answers to the questions 3-5, double up C_n^3 (why?).

Problem 6.10. *Patterns of Chords in a Circle. There are n points on a circle, which split it into n equal arcs $(n\ge 5)$. Each point is joined with a chord with the point next to adjacent to it (all possible chords joining the base points and cutting "double" minor arcs are drawn).*

a) How many chords have been drawn?

b) How many points of intersection of the chords are there inside the circle?

c) How many parts do the chords split the circle into?

d) Imagine that our circle is a billiard table. The billiard balls bounce off the bounds of the table according to the following rule: the angles of incidence and reflection are equal.

A ball is placed in one of the base points and directed along with one of the chords coming from it. Assume that a ball can move infinitely long. Will it always return to its starting point eventually? If not, state the conditions ensuring it always happens. How many chords will the ball pass until it returns in the initial point first?

Answer. a) n; b) n; c) $2n+1$.

Problem 6.11. *Patterns of Chords in a Circle (continued). There are n points on a circle* ($n \geq 7$), *which split it into equal arcs (call these arcs elementary and call the points by base points). Each base point is connected with chords with two base points laying three elementary chords away from it.*

a) How many chords have been drawn?

b) How many points of intersection of the chords are there inside the circle?

c) How many parts do the chords split the circle into?

d) A spider departs from a base point A, and he is going to follow along the designated roads which are the abovementioned chords. How many base points (including A) *will the spider be able to visit?*

Answer. a) n; b) $2n$; c) $3n+1$; d) n *if n is not divisible by 3;* $\frac{n}{3}$ *if n is divisible by 3.*

Solution.

a) Construct the set M, consisting of all pairs $(T;l)$, where T is a base point and l is a chord (from the statement) one of the ends of which is T. Denoting the sought number of chords by x, we will count the number of elements (pairs) of M in two different ways, in order to get the equation for the unknown x. Every base point T is the endpoint of two different chords defined in the statement of the problem. Therefore, the set M contains (the number of base points) $\times 2 = 2n$ pairs. From the other point of view, every chord has two ends, and both of them are base points. Thus, the set M contains (the number of chords)$\times 2 = 2x$ pairs. Hence,

$$2x = 2n,$$

which yields $x = n$.

b) Let A, B, C, D be consecutive base points. The chord AD is crossed only by four chords, which come from the points B and C. Does this mean that there are four points of intersection of the chord AD? What if the chord joining the point C with the point next to A, and the chord connecting the point B with the point next to D, intersect (and they undoubtedly do intersect) in the point on the chord AD? If this is the case, then the chord carries three points of intersection instead of four. So, is such a situation possible? Can three of our chords intersect in one point? To answer this question, we need to take into account all circumstances provided by the statement of the problem.

First, all chords are of the same length. Is it possible to draw three equal chords through one point in a circle? No, unless the point is the center of a circle. The center of a circle is an exception.

Second, is it possible that our chords are diameters? Yes, but only if $n = 6$. Bu the statement of the problem reads $n \geq 7$.

The conclusion. Three of our chords can not intersect at one point. Therefore, there are 4 points of intersection of chords on the chord AD. And all chords are equal in this context. Thus, there are four points of intersection on all other chords as well. The number $4n$ is

twice the amount of points of intersection because every such point belongs to two chords. Hence, the answer is $2n$.

c) Imagine that we draw a chord after chord and observe how each next of them impacts the number of parts which the circle is split into. Let several chords (it does not matter which) have already been drawn. They split the circle into a certain number of parts, say, s. How will this number increase if we draw the next chord? It is easy to see that the answer, in general, depends on the exact chord that we are about to draw. In particular, the more of the available chords it crosses, the greater the number s changes. Let us take a careful look at this dependence. To this end, imagine the process of the creation of the new chord. The tip of our pencil moves slowly from one end of the future chord to the other. It leaves the line behind it, which begins to split the previously solid part of the circle into two parts. As this extending ray reaches any of the previously drawn chords, the process of splitting ceases, and the number s increases by 1. The ray moves ahead slowly but truly. Now, it begins splitting another part of the circle, and once it reaches another chord, the aggregate amount of parts increases by one again. This amount increases by 1 for the last time when the point (the tip of the pencil) drawing the new chord reaches its endpoint. The interim conclusion: when we draw a new chord, the number of parts of the circle increases by $t+1$, where t is the number of points of the intersection of this new chord with the chords that have been drawn previously.

This immediately yields the final result: if we draw all chords in question, then they split the circle into $3n+1$ parts ($n+2n+1 =$ (the number of chords) + (the number of points of intersection of chords) + (one part (the entire circle) which was available before any chords).

Problem 6.12. *Patterns of Chords in a Circle (the generalization of two previous problems). There are n points on a circle ($n \geq 2k+3$, k is given natural number), which split it into n equal arcs (call these arcs elementary and call the points by base points). The chords are drawn which satisfy the following conditions:*

(I) their endpoints are the base points;

(II) the minor arc corresponding to each chord contains k more base points.

a) How many chords have been drawn?

b) How many points of intersection of the chords are there inside the circle?

c) How many parts do the chords split the circle into?

d) Let us call two base points A and B allied if one can get from one of them to the other moving along the above chords. The set of allied points we call a family. How many families do the chords split the set of base points into? Find the number of points in each family.

Answer. a) n; b) kn; c) $(k+1)n+1$; d) One family if n is not divisible by $k+1$; $k+1$ families if n is divisible by $k+1$. In the first case, the family comprises all the base points, and in the second, each family contains $\frac{n}{k+1}$ points. Geometrically, each family is the set of all vertices of a closed polygonal chain, the segments of which are chords that are drawn as prescribed by the statement.

Problem 6.13. *There are 24 (base) points on a circle, which split it into equal arcs.*

a) How many different (by the numbers of their vertices) regular polygons have all their vertices in the base points?

b) How many regular polygons have all their vertices in the base points?
Answer. *a) 6; b) 24.*

Problem 6.14. *(A generalization of the previous problem). There are n (base) points on a circle, which split it into equal arcs.*

Assuming that
$$n = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$$
is the prime factorization of the number n ($p_1, p_2, ..., p_s$ different prime numbers none of which is 2; $k_1, k_2, ..., k_s$ are natural numbers), answer the following questions:

a) How many different (by the numbers of their vertices) regular polygons have all their vertices in the base points?

b) How many regular polygons have all their vertices in the base points?

How will the answers to questions a) and b) change if
$$n = 2^{k_0} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_s^{k_s}$$
(k_0 is natural)?

Answer. *a)* $(k_1+1)(k_2+1)...(k_s+1)-1$; *b)* $\frac{p_1^{k_1}-1}{p_1-1} \cdot \frac{p_2^{k_2}-1}{p_2-1} \cdot ... \cdot \frac{p_s^{k_s}-1}{p_s-1} - n.$

Solution. a) In order for k base points to be the vertices of a regular k-gon, they must split the circle into equal arcs, and this is only possible if k is the divisor of n. In addition, according to the definition of the polygon, k can not be equal to 1 or 2. Finally: the amount of different regular polygons having all their vertices in the base points is the same as the amount of different natural divisors of n excluding 1 and 2. Thus, we have a clearly arithmetical problem which is equivalent to the original geometrical one. Here is this problem:

How many natural divisors of the number n are there if the prime factorization of n is
$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_s^{k_s}?$$
In the above equality, different symbols p_i denote different prime numbers, and the letters k_j denote some natural numbers.

The answer to this essential arithmetical-combinatorial question bases on the uniqueness of the prime factors decomposition of any natural number. In order for a number m to be the divisor of the number n, it is necessary for it not to have any other prime divisors except for those that are available in the prime factorization of n. As to the prime divisors of n, there should not be more instances of each of them in the prime factorization of m, then there are in the prime factorization of n. In other words, any divisor m of the number n can be expressed as
$$m = p_1^{l_1} \cdot p_2^{l_2} \cdot ... \cdot p_s^{l_s}, \tag{6.18}$$
provided that the additional conditions
$$0 \le l_i \le k_i \ \ (i = 1, 2, ..., s) \tag{6.19}$$
hold. Conversely, any number (6.18) for which conditions (6.19) hold is the divisor of the number
$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_s^{k_s},$$

because
$$p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s} = (p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_s^{l_s}) \cdot (p_1^{k_1-l_1} \cdot p_2^{k_2-l_2} \cdot \ldots \cdot p_s^{k_s-l_s}),$$
and the number in the second pair of parentheses is integer.

It remains to determine how many numbers (6.18) satisfying (6.19) exist.

We can not manipulate the numbers p_i: accompanied by the powers k_i they create strict limitations dictated by the number n. However, we can vary the powers l_i provided that conditions (6.19) hold. The amount of divisors of the number n is the same as the amount of different sets of s numbers

$$(l_1, l_2, l_3, \ldots, l_s) \tag{6.20}$$

which satisfy conditions (6.19). There are $k_1 + 1$ (0 to k_1) possible values for the first component l_1, $k_2 + 1$ for l_2, and so on, finally, there are $k_s + 1$ possible values for l_s. In addition, the values of different components can be combined randomly. Thus, by virtue of the combinatorial rule of product, the amount of different sets (6.20) satisfying conditions (6.19) is
$$(k_1+1)(k_2+1)\ldots(k_s+1).$$
Hence the amount of divisors (6.18) of the number n.

For instance, let us present the full list of sequences (6.20) and the corresponding divisors of n for the case $n = 24$.

We have: $24 = 2^3 \cdot 3$. The formula for divisors: $m = 2^{l_1} \cdot 3^{l_2}$, $0 \le l_1 \le 1$, $0 \le l_2 \le 1$.

The table 6.4 gives the list of divisors for the number 24:

Table 6.4. List of divisors of the number 24

l_1	l_2	m
0	0	1
0	1	3
1	0	2
1	1	6
2	0	4
2	1	12
3	0	8
3	1	24

Total amount of divisors is $(3+1)\cdot(1+1) = 8$.

Let us get back to the question about regular polygons.

If there is no number 2 among the prime divisors p_i of the number n (the number n is odd), then there are
$$(k_1+1)(k_2+1)\ldots(k_s+1) - 1$$
different regular polygons with all vertices in the base points.

Alternatively, if
$$n = 2^{k_0} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s},$$

then there are

$$(k_0+1)(k_1+1)(k_2+1)\ldots(k_s+1)-2$$

polygons.

In the special case of $n=24$, there are $8-2=6$ polygons with different numbers of sides, namely: triangle, rectangle, hexagon, octagon, 12-gon and 24-gon.

b) If m is the divisor of n and $m \geq 3$, then, as we already know, there exists a regular m-gon with vertices in the base points. Moving it around the circle by the angle $\frac{360^0}{n}$, we get another m-gon, the vertices of which also lay in (other) base points. The same concerns to the turns of the original m-gon by the angles

$$\frac{360^0}{n}\cdot 2,\ \frac{360^0}{n}\cdot 3, \ldots, \frac{360^0}{n}\left(\frac{n}{m}-1\right).$$

Thus, we get $\frac{n}{m}$ different regular m-gons with vertices in the base points.

If the number m is prescribed all possible values of the divisors of the number n excluding 1, then the number $\frac{n}{m}$ gains various values of the divisors of n, except for n itself. Therefore, the aggregate amount of regular polygons in question equals to the sum of all positive divisors of n, except for n itself (we emphasize that we consider the case of odd n here).

For example, in the case $n=45$, there are

$$1+3+5+9+15=33$$

regular polygons. The complete list of them is presented below:

one 45-gon;

three 15-gons;

five 9-gons;

nine pentagons;

fifteen triangles.

Needless to say, if n is even, then the amount of regular polygons is defined by the amount of divisors of n excluding n itself and $\frac{n}{2}$.

For example, for $n=24$, we have

$$1+2+3+4+6+8=24$$

polygons. The full list follows:

one 24-gon;

two 12-gons;

three octagons;

four hexagons;

six rectangles;

eight triangles.

We have related the amount of regular polygons in question to the sum of divisors of n decomposed into prime factors:

$$n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots\cdot p_s^{k_s}.$$

The next question is how to express the sum of divisors of n with the numbers p_i and k_j?

Consider the product of s sums:

$$\begin{aligned}&(1+p_1+p_1^2+p_1^3+\ldots+p_1^{k_1})\times\\&\times(1+p_2+p_2^2+p_2^3+\ldots+p_2^{k_2})\times\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&\times(1+p_s+p_s^2+p_s^3+\ldots+p_s^{k_s}),\end{aligned}$$

and imagine the result of the above multiplication (the expression that is the result of the removal of all parentheses). And the result would be nothing else but the sum of all divisors of the number n, as any summand would be of the form:

$$p_1^{i_1}\cdot p_2^{i_2}\cdots p_s^{i_s}.$$

In addition, $0\le i_1\le k_1$, $0\le i_2\le k_2$, ..., $0\le i_s\le k_s$, and every such set of values for $i_1, i_2, \ldots, i_s$ have one summand corresponding to it. That is why the sum of all divisors of n equals to

$$\frac{p_1^{k_1+1}}{p_1-1}\cdot\frac{p_2^{k_2+1}}{p_2-1}\cdots\frac{p_s^{k_s+1}}{p_s-1}.$$

Chapter 7

Trees

In this chapter, we consider the combinatorial aspects of a certain type of so-called undirected graphs. More precisely, we consider those finite graphs which are called trees.

1. Assume that we are provided with a finite collection (a finite set) of points. We can assume these points to be fixed (selected) points of the usual three-dimensional Euclidean space denoted, say, by the letters $A_1, A_2, A_3, \ldots, A_n$. However, the latter assumption is unnecessary. The points can be deemed to be letters themselves or any other objects denoted by them. Whichever we choose, this will not impede the successful introduction of the notion of the graph.

So, there is a set of points $A_1, A_2, \ldots, A_n$. When we interpret them as geometrical objects, it is easy to imagine that some of them are connected with line segments. Such geometrical construction, the set of fixed points along with the set of line segments joining some of these points, is called an undirected finite graph. The points $A_1, A_2, \ldots, A_n$ are called vertices (or nodes) of a graph, and the line segments are called its edges (lines, arcs). If an edge connects the vertices A_i and A_j, then it may be denoted by A_iA_j, or equivalently, A_jA_i, similarly to the way in which line segments are usually denoted in geometry. In this case, it can also be said: there exists an edge leading from A_i to A_j (and vice-versa).

If the vertices of a graph are symbols $A_1, A_2, \ldots, A_n$, then the edges of this graph are pairs of symbols $(A_i; A_j)$. The pairs $(A_i; A_j)$ and $(A_j; A_i)$ are considered to be equal as they are associated with the same edge. Sometimes, to denote an edge $(A_i; A_j)$, it is convenient to introduce other symbols, which are not directly related to the denotation of the vertices joined by this edge.

If a vertex A_i of a graph is joined with edges with other k vertices of this graph, then the number k is called the degree (or valency) of this vertex. The degree of any vertex of a graph is an integer from the interval $[0, n-1]$, where n is the number of vertices of a graph. Vertices with degree 0 (zero) are not joined with any other vertex of a graph. They are called isolated vertices. If all vertices of a graph are isolated, then there are no edges in such a graph. Another extreme case is where every vertex of a graph has the maximum degree $n-1$ (is joined with edges with all other vertices). Such a graph is called a complete (or total) graph.

In Fig. 7.1, there is a graph with 8 vertices and 6 edges (it is not unusual to depict the edges of a graph with non-straight lines; a line of any type leading from one vertex to another, means that these vertices are joined with an edge). Its vertex A is isolated, the

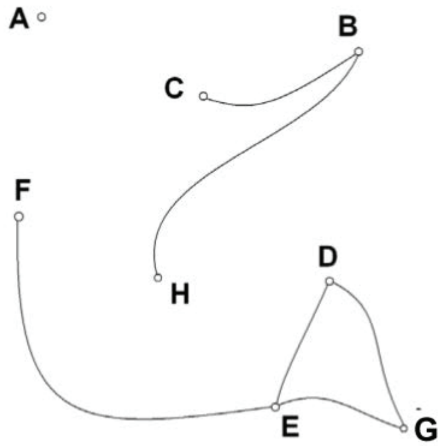

Figure 7.1. Graph with 8 vertices and 6 edges.

vertices C, H and F have a degree 1, the vertices B, D and G have degree 2, and the vertex E have degree 3.

Two edges of a graph are called adjacent if they are separated by one vertex. Another equivalent definition is the following: adjacent edges are those that share a vertex. In Fig. 7.1, the edges CB and BH are adjacent. Other pairs of adjacent edges include FE and ED, FE and EG, EG and GD, ED and DG, DE and EG.

Vertices A_i and A_j of a graph are called adjacent if there is the edge A_iA_j in a graph. For example, the graph is shown in Fig. 7.1 has the following pairs of adjacent vertices: B and C, B and H, F and E, E and D, E and G, D and G. On the other hand, the vertices A and B, or A and G, or C and, etc., are not adjacent.

The sequence $T_1T_2...T_s$ of the vertices of a graph is called a chain (or walk, path) of vertices if any two neighboring vertices in this sequence are adjacent vertices of a graph. A chain is called closed if its last vertex coincides with the first one. A closed chain is usually referred to as a tour. Several examples of chains of vertices of the graph are shown in Fig. 7.1 : CBH, HCB, CBC, $FEDG$, GED, $FEGDE$, $EDGE$. The first and the last of these chains are closed (and as such are tours). An open chain is called simple if none of its vertices repeat. A tour is called simple (or a cycle) if none of its vertices repeat, except for the initial and the last one, which are equal by definition. For example, CBH, $FEDG$ and BC are simple open walks, and $EDGE$ is a cycle in Fig. 7.1.

An open chain $T_1T_2...T_s$ is said to connect the vertices T_1 and T_s. The existence of such chain evidences that one can walk along the edges T_1T_2, T_2T_3, ..., $T_{s-1}T_s$ from the vertex T_1 to the vertex T_s (and in the reverse direction). If there is a chain in a graph that joins vertices A and B, then there exists a simple chain joining these vertices. This fact can be proved as follows. Let us take any chain joining (different) vertices A and B. If it is simple, then the proof is complete. Otherwise, some of its vertices repeat several (2 or more) times. Consider any part $X\ X$ of this chain which is a tour. Then the entire chain is of the form:

$$A...EX...XF...B.$$

Removing the part $X\ X$ from it and substituting it with the vertex X, we get the chain that joins A and B again. Here is this chain:

$$A...EXF...B.$$

This is a chain and not just a sequence of vertices, because E and X, as well as X and F are pairs of adjacent vertices, which is evidenced by their positions in the previous chain. The new chain is shorter than the previous one by at least two positions (vertices). It appears that any non-simple chain can be shortened (reduced, contracted) by at least two positions. If the new chain is still non-simple, then the procedure of contraction is repeated again, and so on. However, such a process can not last infinitely as the original chain has a finite number of positions. So, eventually, we will get a simple chain.

The notion of a chain is closely related to another essential characteristic of a graph, which we describe with the following example first. Observing the graph in Fig. 7.1, one notices an eye-catching peculiarity: it is constructed of three isolated parts. The first part consists of the vertices F, E, D, and G along with the corresponding edges, the second comprises the vertices B, C and H and the edges BC and BH, and the third part is the only vertex A. These parts are called connected components (or just components) of a graph. Which feature of any component is underlined by the above name? Any two vertices of a component can be connected to each other by a path along its edges. In other words, if K is a connected component (comprising more than one vertex), A and B are two of its vertices, then a chain joining A and B can be constructed of the elements (edges and vertices) of K. However, there is an additional aspect that needs to be accounted for in the definition of a connected component. This is the maximal property of the set of vertices composing a component. If K is a connected component, then any vertex C of the graph that does not belong to this component (if any) should be completely isolated from it. This means that there is no edge in a graph connecting a vertex from K with the vertex C.

Which considerations can be taken into account to ensure that a graph consists of several (or, possibly, one) connected components? Take an arbitrary vertex A of a graph. The graph either has vertices connected with it or not. In the latter case, the vertex A comprises a separate component. As to the former case, we classify to the set K the vertex A as well as all vertices which can be joined with it by a chain. The set K constructed in the above manner is a connected component, which is evidenced by the following considerations. Let P and Q be two vertices from K. According to the construction of K, there exist chains $PT_1...T_jA$ and $AS_1S_2...S_iQ$ which connect P with A and A with Q. Combining them, we get the chain $PT_1...T_jAS_1...S_iQ$ that connects P to Q. Therefore, any two vertices of the constructed set K can be connected with a chain. Now, let $B \in K$ and $C \notin K$. We have to ascertain that there is no edge between B and C, that is, that these vertices are not adjacent. Let us prove it by contradiction. Assume, B and C are joined with an edge, that is there exists a chain BC. The structure of the set K evidence that there exists a chain $AL_1...L_tB$ joining A and B. Extending this chain with the edge BC, we get the chain $AL_1...L_tBC$ that connects A to C. But this means that the vertex C has belonged to the set K from the very beginning, which contradicts our assumption about it. Therefore, the set K satisfies both the condition of connectivity and the condition of maximality, which proves that K is a connected component. Now, there are two possibilities: either K coincides with the entire graph or not. In the former case, the graph has one component. Such a graph is called

a connected graph or one-component graph. In the latter case, for any vertex A' which does not belong to K, we outline the component that includes it, following the procedure implemented above for the vertex A. Thus, step by step we consider the entire graph. So, indeed, a graph is composed of several connected components.

There are several remarks that are to be made concerning closed chains (tours). The most general and absolutely natural definition of a tour (it is presented above – it is a chain that begins and ends with the same vertex) is overly general and as such, is of little interest. In particular, it does not highlight any peculiarities of graphs. Any graph which has edges (at least one) has tours as well. Really, if AB is an edge, then ABA is a tour; if $ABCD$ is a chain, then $ABCDBA$ is a tour. It appears that our definition results in a way to some vertex and back along the same edges being considered as a tour, which is not consistent with the usual view on this concept and does not reveal any information about the structural specifics of a graph. This shortcoming is inherent to two-edge cycles ABA as well. However, other cycles (involving three or more vertices) fall in line with our perception of closed round paths. The existence or absence of such walks in a graph is definitely among its essential characteristics. Therefore, further we will just say “cycle” instead of the phrase “cycle of length 3 or more”.

The presence of a cycle in a graph which passes through vertices A and B provides that these vertices can be connected with two different simple chains (“different” here means that these chains with endpoints in A and B do not have shared inner vertices). Indeed, let $T_1T_2 \ldots T_sAP_1P_2 \ldots P_iBQ_1Q_2 \ldots Q_jT_1$ be a cycle. Then

$AP_1P_2 \ldots P_iB$ and $AT_s \ldots T_2T_1Q_j \ldots Q_2Q_1B$

are two different simple chains connecting A and B.

Conversely, if we can construct two different simple chains from the vertex A to the vertex B, then there is a cycle that passes through these two points. For example, let $AP_1P_2 \ldots P_iB$ and $AQ_1Q_2 \ldots Q_jB$ be two different simple chains that connect the vertices A and B. Then $AP_1P_2 \ldots P_iBQ_j \ldots Q_2Q_1A$ is a cycle that walks through these two vertices.

As a conclusion of the above several paragraphs, there come the following facts. If it is known that there are no tours (round paths) in a graph, then any two of its vertices either are isolated from each other (belong to different connected components) or can be joined with a unique simple chain. In particular, if a graph is a connected one and it does not have cycles (connected acyclic graph), then any two its vertices are the endpoints of the one and only chain (in other words, there is only one way to get from one of its vertices to any other along its edges). Conversely, if there is a unique path leading from any of its vertices to any other vertex, then we have a connected graph with no cycles.

2. A connected acyclic graph is called a tree. From the previous paragraph, we conclude that any two vertices of a tree are joined with one and only a simple chain. Below, we outline several other properties of trees.

a) A tree of order ≥ 2 (order is the number of vertices) has at least one vertex with degree 1 (such vertices are called pendant vertices).

One way to ensure that this property holds is the following. Let A_1 be an arbitrary vertex of a tree. We depart from this vertex moving along the edges of the tree and adhering to the rule: having entered any vertex A_k, we exit it following a new edge and never following the one which we have used to get to it. Moving in such a manner, we will never visit the same vertex P twice. Indeed, if this was the case, then the part of our path between the first and

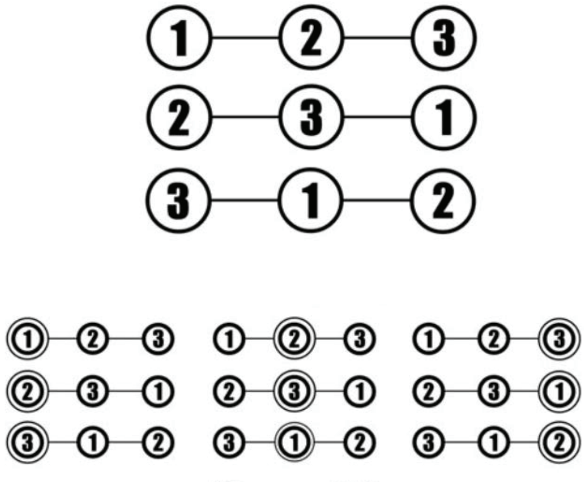

Figure 7.2. 3 regular and 9 rooted trees.

the second visits to the point P would have been a cycle, which is prohibited in a tree. Thus, we pass new and new vertices on our way. However, this journey can not last infinitely as any graph has a finite number of vertices. Eventually, there will be a vertex A_s that puts an end to our trip. And this very point will be the one with degree 1. Really, this vertex is the endpoint of the edge that led us to it. There is no other edge incident with it (adjacent to it) because if there were one, we could have continued our trip.

b) If there are n vertices in a tree, then there are $n-1$ edges.

Let D_n be a tree with n vertices and m edges. We have to prove that $m = n-1$. Contemplate as follows. The tree D_n has (at least one) pendant vertex. Removing it along with an edge incident with it, we get a tree D_{n-1} which has one vertex and one edge less than D_n. Why D_{n-1} is necessarily a tree (and not a graph that does not comply with the definition of a tree)? When we remove a leaf vertex and an edge joining it with an adjacent vertex, the connectivity of the graph does not fail. There are no new cycles in the resulting graph as well. Hence, D_{n-1} is actually a tree. Obviously, there are $n-1$ vertices and $m-1$ edges in it.

We apply the same procedure to the tree D_{n-1} to get a tree D_{n-2}, which has $n-2$ vertices and $m-2$ edges. Having repeated this procedure $n-1$ times, we get a tree D_1 that consists of one vertex and $m-(n-1)$ edges. But a tree with one vertex has no edges at all. Hence, $m-(n-1) = 0$, that is $m = n-1$.

c) sometimes, a vertex is distinguished to be called a root. In this case, a tree is called a rooted tree. If a tree is composed of n vertices $A_1, A_2, \ldots, A_n$ (connected with $n-1$ edges in order to produce a connected graph), then any of them can be declared a root. This implies that there are n times more rooted trees than there are "regular" ones, given the trees of both types are constructed with the same n vertices. For example, basing on three vertices (denote them with $1, 2$ and 3 instead of A_1, A_2 and A_3 for our convenience), there can be constructed 3 regular and 9 rooted trees Fig. 7.2.

d) Let D be a tree with the root A. We already know that there exists only one chain

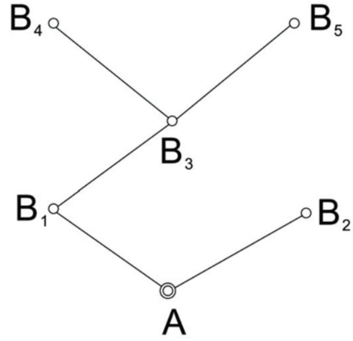

Figure 7.3. Tree with the root A.

that connects A to B. If its length (the number of edges in it) equals k, then we say that there is the distance of k between the root and the vertex B. In Fig. 7.3, there is a tree with the root A. Its vertices B_1, B_2, B_3, B_4 and B_5 are distanced from the root by $1, 1, 2, 2$, and 3 respectively. There is a partition of all vertices of a tree into subsets by their distance from the root, which is called the level structure of a graph. The vertices from the same subset are called the vertices of the same level. Zero level is the lowest one. There is only one vertex in this level – the root itself. It comprises the class K_0. The class K_1 is composed of all vertices that are distanced by 1 from the root of the tree. These are the vertices of the first level. They always exist (at least one), unless a tree has just one vertex. If there were none, then the root of a tree would be an isolated point that contradicts one of the defining properties of the tree – its connectivity. We assign to the class K_2 those vertices of a tree which is in distance 2 from the root, and so on. A vertex from the class (subset) K_s and a level s vertex are two names of the same vertex. Both define a vertex laying in distance s from the root.

If there is at least on level s vertex in a tree (that is, the set K_s is non-empty), then there are vertices of all lower levels in it, from zero to the $(s-1)$-th. In order to verify it, it suffices to imagine a chain connecting the root of a tree A with the vertex B. It contains $s-1$ interim vertices $B_1, B_2, ..., B_{s-1}$, which belong to the classes $K_1, K_2, ..., K_{s-1}$ respectively. This property can be put in another way: any level s $(s > 0)$ vertex is connected with an edge with one (and only) vertex of the level $s-1$. There is only one way to "slide" along an edge from a level s vertex to a vertex of level $s-1$.

As to the transition from level s to level $s+1$, there are different options. If a vertex of level s is a pendant vertex, then there is no way to "climb" to the level $s+1$ along an edge of a tree. Alternatively, if this vertex is not a pendant one, then there is at least one way to ascend to the level $s+1$ from it (though not necessarily the only way).

The properties of a rooted tree presented in two previous paragraphs can be summarized as follows: each vertex (except the root) of a rooted tree has only one descending edge incident with it; any vertex (except for pendant vertices) of a rooted tree has at least one ascending edge incident with it. If there are two or more ascending edges adjacent to some vertex, then such vertex is called a branching point of a tree. If we walk along the edges of a tree "top to bottom", departing from the root and moving to the higher level vertex than the

previous one in each step, then we can influence our route only when we reach branching points.

e) The chains which connect the root of a tree with its pendant points cover the entire tree. This statement should be interpreted as follows: every vertex of a tree belongs to at least one chain that connects its root to a pendant vertex. Really, let B be any non-root vertex of a tree. Construct a chain joining the root A with B. If this chain can not be extended, then this evidences that the vertex B is a pendant vertex, and the constructed chain is the sought one. Otherwise, we extend the chain $A...B$ until we get to a pendant vertex. Considerations presented in section a) imply that this eventually happens.

The chains that connect the root with pendant vertices are called maximal chains. They are not maximal in the sense of their lengths (there can be chains of different lengths among them), but because of the fact that they can not be extended.

According to the above definition, any two maximal chains have the same starting point, which is the root of a tree. Can they have other shared points? The tree shown in Fig. 7.3 advises that the answer is positive. The maximal chains $AB_1B_3B_4$ and $AB_1B_3B_5$ have shared vertices A, B_1 and B_3. These vertices create a chain which is the initial part of the chains $AB_1B_3B_4$ and $AB_1B_3B_5$. This property is intrinsic to any two maximal chains of a rooted tree. Any two maximal chains start with a shared part (which itself is a chain), then branch and share no points further. This is caused by the fact that any vertex of a rooted tree is connected to the root with the only chain. For example, let B be a shared vertex of maximal walks W_1 and W_2 of a tree with the root A. Then the walk W_1 is the concatenation of two chains:

$A...B$ and $B...C_1$,

where C_1 is the ending point of the walk W_1. Similarly, the chain W_2 is the concatenation of the chains

$A...B$ and $B...C_2$,

where C_2 is the ending point of the chain W_2. It appears that all vertices of the chain $A...B$ are shared by the chains W_1 and W_2.

f) In order to bridge the gap between the terminology concerned with rooted trees as a special case of graphs and trees in a casual sense, we introduce alternative names of two of the above notions. We will call pendant vertices of a rooted tree leaf vertices. The tree shown in Fig. 7.3 has three leaf vertices: B_2, B_4 and B_5.

We will call a chain that connects the root or any branching vertex with a leaf vertex a branch. Here is the complete list of all branches of the tree depicted in Fig. 7.3: $AB_1B_3B_4$, $AB_1B_3B_5$, B_3B_4, B_3B_5 and AB_2. As we can see, a branch may be a part of another branch. Any maximal walk is a branch. It is appropriate to call these branches trunks, or trunk branches (they are also called root-to-leaf paths). All other branches are incidental. Any incidental branch is a part of a root-to-leaf walk. In the tree in Fig. 7.3, $AB_1B_3B_4$, $AB_1B_3B_5$ and AB_2 are trunk branches, while B_3B_4 and B_3B_5 are incidental branches.

We will call two branches of a tree unrelated if they do not have shared vertices or share only one vertex. In the latter case, the shared vertex is always the initial vertex of one or both of the branches. Indeed, if two branches shared a vertex B that is interim for both, then they would share a vertex C that is connected with an ascending (w.r.t. C) edge with the vertex B. Thus, these two branches would have at least two shared vertices. Therefore,

two unrelated branches either have a common starting point in a branching vertex or one of them is an “excrescence” of the other. Here are several pairs of unrelated branches of the tree in Fig. 7.3:

$AB_1B_3B_4$ and B_3B_5; AB_2 and B_3B_4; AB_2 and $AB_1B_3B_5$; B_3B_4 and B_3B_5. There are three more such pairs. Find them.

If a tree has k leaves (leaf vertices), then there exist k pairwise unrelated vertices which compose the entire tree. There is another formulation of this assertion: every tree that has k leaves can be decomposed into k pairwise unrelated branches. Before getting down to the proof of this property of rooted trees, let us present several examples, again with the help of Fig. 7.3. The tree shown in this Figure has three leaves. Here are two possible decompositions of this tree into three pairwise unrelated branches: 1) $AB_1B_3B_4$, B_3B_5 and AB_2; 2) $AB_1B_3B_5$, B_3B_4 and AB_2.

Now, let us prove the above assertion. So, let us have a rooted tree with leaves $C_1, C_2, ..., C_k$ and the root A. Consider any leaf, say, C_1 and construct a descending chain from it to the root A. We already know that this chain exists, and in addition, it is unique. Obviously, the constructed chain is a root-to-leaf branch. Let us turn to the vertex C_2 now (or any vertex other than C_1). We “descend” from it along the edges of the graph until we get to a vertex that belongs to the previously constructed branch. Let it be a vertex T_2 (we will inevitably reach such vertex on our way down from C_2; in the case of “the longest” descend we get to the root A). Thus, we get the branch $T_2...C_2$, which is unrelated to the branch constructed in the first step, as it shares only the vertex T_2 with it. Now, we begin the descent from the leaf C_3, which lasts until we reach the vertex T_3 that belongs to the union of the previously constructed branches. The branch $T_3...C_3$ is unrelated to any of the previously created branches because it has only one shared point with their union. Proceeding in this fashion, in k steps we will have k pairwise unrelated branches. They exhaust the entire tree in the sense that every vertex of the tree belongs to at least one of the constructed branches. In order to prove this, we outline the special property of the constructed “bunch” of k branches. If $T_s...C_s$ is one of these branches and T_s is not the root, then T_s belongs to the branch $T_m...C_m$ from this bunch, with T_m being the vertex of the lower level than T_s. This means that if we extend (to the bottom) any non-root-to-leaf branch of the constructed bunch up to the moment when it reaches a root-to-leaf brunch (and there is the only way to make this), then all vertices of this root-to-leaf branch belong to our bunch. After that, turn to the final chord of the proof. Extending all branches of the constructed bunch up to root-to-leaf branches, we get all root-to-leaf branches (maximal paths) with no exception, as their amount is equal to the amount of leaves. It remains to recall that any vertex of a tree always belongs to at least one maximal chain.

3. In the current and subsequent sections, we will search the answer to the question: how many different rooted trees with n vertices are there? Taking a closer look at the essence of the above question, one finds out that it requires a certain clarification. To answer it, we have to agree upon the definition of different trees. There are two major options here.

Consider Fig. 7.4. Are the two trees in it different or not? They have the same configuration and only differ in the way their vertices are named. If we rename the vertices of the left tree by the rule

$$A \to C,\ B \to A,$$
$$C \to B,\ D \to D,$$

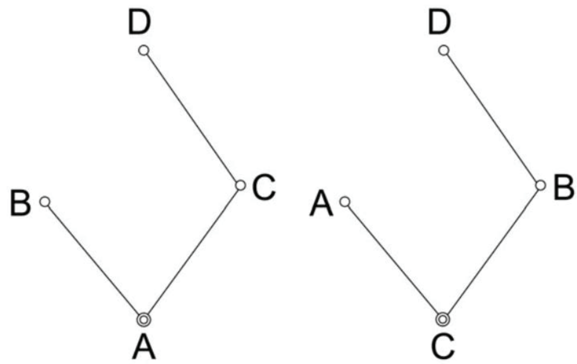

Figure 7.4. Two trees.

then we get the right tree. This is because the above rule that governs the renaming of vertices is consistent with the placing of edges in both trees. For example, there is the edge AB in the left tree, which in addition, has the special endpoint A (the root of the tree). Renaming $A \to C$, $B \to A$, we get a similar property that concerns the right tree. It has the edge CA, the endpoint C of which is special again (and this vertex is the root of the right tree).

The trees of the type shown in Fig. 7.4 are called isomorphic. Below we provide the exact definition of this notion. Let there be two trees with the same numbers of vertices. Let the first tree D_1 be constructed on the set of vertices $P = \{A_1, A_2, ..., A_n\}$ and A_1 is its root, and the tree D_2 be constructed on the set of vertices $Q = \{B_1, B_2, ..., B_n\}$ with B_1 being its root. The trees D_1 and D_2 are called isomorphic if a bijection

$$P \leftrightarrow Q,$$

can be established between the sets of their vertices P and Q, such that the roots A_1 and B_1 correspond to each other, and this bijection is consistent with the placing of edges in both trees. The latter condition has the following meaning. If according to the abovementioned bijection, the vertices A_k and A_s correspond to the vertices B_i and B_j respectively, then both edges A_kA_s and B_iB_j are either available in the respective trees or absent in both trees. A bijection between the vertices of two trees that has the above properties is called an isomorphic bijection or isomorphism.

If we agree to assume that isomorphic trees are identical disregarding the names of their vertices, then the phrase "different trees" mean "non-isomorphic trees" for us.

Besides, if the phrase about the correspondence of roots is removed from the definition of isomorphic trees, then it becomes the definition of isomorphic (unrooted) trees. Moreover, the notion of isomorphism becomes applicable to arbitrary graphs (not necessarily trees).

The rooted trees in Fig. 7.4 are isomorphic. Hence, assuming "different equals to non-isomorphic" these trees are identical.

However, a completely different point of view exists. Imagine the set of certain objects that have specific individual characteristics which distinguish them (they can be points,

symbols, etc.) from each other. Combining some of these objects in pairs (whereas the same object can be a component of several pairs), we create a graph. The given objects are its vertices and the created pairs are the edges. Comparing two graphs constructed on the same sets of vertices, not only we can be interested in their structure but also whether the same vertices sit in the same places in both graphs. We can only consider two graphs to be identical if every vertex (we emphasize that we clearly distinguish any vertex from all others disregarding various configurations that it creates combined with other vertices) is in the same position in both graphs. If at least one vertex has a different place in a graph Γ_1 than it has in a graph Γ_2, then the graphs Γ_1 and Γ_2 are considered different. This approach to the comparison of graphs is completely different from the previous one. To distinguish these two approaches, let us agree that in the first case we will call two graphs isomorphic or non-isomorphic, while in the second case we will claim that they are identical or different (non-identical). The above concerns trees and rooted trees as well, because the trees are a special case of graphs.

The trees shown in Fig. 7.4 are isomorphic but non-identical, because, for example, they have different roots. Considered as non-rooted trees, they are still non-identical, because, for example, the vertex A is of degree 2 in the left tree, while it has degree 1 in the right.

4. The graphs in general, as well as their special types (e.g., trees), are the subject of many combinatorial problems, including the instructive ones in terms of the methodology applied to their solution and important for practical applications. If one was about to compile a handbook of such problems which should begin with the most general ones, then it would start with the problems of the following types:

How many different graphs are there on n given vertices?

How many non-isomorphic graphs are there on n given vertices?

How many different trees are there on n given vertices?

How many non-isomorphic trees are there on n given vertices?

How many different rooted trees are there on n given vertices?

How many non-isomorphic rooted trees are there on n given vertices?

The first of the above problems is rather straightforward. At least, we have faced much more complicated problems in the previous chapters. Therefore, we include it in the problems section of this chapter and strongly encourage the readers to solve it themselves.

On the contrary, the second problem is so challenging that we will not even discuss it in the general case. However, we will analyze the situation in the case of a 4-vertex graph to appreciate the reason for the second problem to be incomparably tougher than the first one. As at the first glance, these two problems may seem related and close to each other by their context, which may create an illusion that the methods of their solving are similar or, at least, differ insignificantly. In fact, it is not true. While the first problem, as we have mentioned above, is a routine one, the second one is extremely challenging.

So, let there be 4 vertices denoted by the letters a, b, c, and d. How many different graphs can be constructed by joining some of them (and not joining the others) with edges? We can omit edges as such and get a graph in which every vertex is isolated from the three others as if every vertex is on its own. This graph is unique in the sense that there are no graphs isomorphic to it (see Fig. 7.5 (1)).

Allow a single edge, and we immediately get quite many graphs. This edge can join any two of the available four vertices. There are C_4^2 (which equals 6) options to choose the

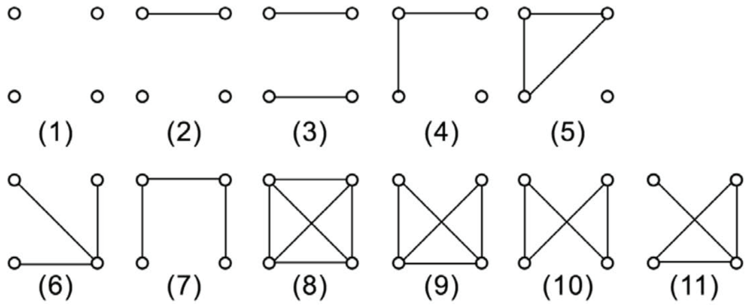

Figure 7.5. Many graphs.

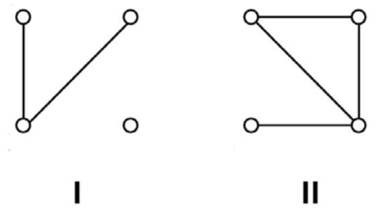

Figure 7.6. Dual graphs.

vertices to be joined. Thus, there are 6 different graphs, which are structurally equivalent, and as such are isomorphic (see Fig. 7.5 (2)).

The graphs with two edges are not all isomorphic. Three of them have the structure characterized by two pairs of vertices joined with edges (which reminds of two dumbbells). This type of structure is shown in (see Fig. 7.5 (3)). Twelve more graphs have one isolated vertex and three vertices joined sequentially by two edges (see Fig. 7.5 (4)).

Let us move on to the graphs with three edges. A brief drawing problem that involves sketching the required graphs (drawing four circles that denote the vertices and three line segments that join some of these circles with each other), evidence that there are three essentially different structures. They are shown in Fig. 7.5 enumerated by (5), (6), and (7). In other words, there are three types of graphs with three edges from the point of view of isomorphism. Putting it another way: such graphs comprise three groups (classes); all graphs of the same group are isomorphic to each other and the graphs from different groups are not. Structures (5), (6), and (7) depict these classes geometrically, providing information about the types of graphs in the respective classes. It remains to remark that there are 4 different graphs in classes (5) and (6) each, and there are 12 graphs in class (7).

The following notion will help us to conclude the brief overview and classification of graphs on four vertices. Let us call two graphs (constructed on the same vertices) dual to each other if in places where one of them has an edge the other does not. For example, the graphs shown in Fig. 7.6 are dual to each other. We emphasize that the notion of duality concerns arbitrary graphs and not only to those on 4 vertices.

The notion of duality is easily and naturally transferrable from individual graphs to the structures, that is to the classes of isomorphic graphs. Consider the structure of a graph, the edges of which are depicted with blue lines. If we draw all missing edges with red lines and then remove all blue lines, then we get the structure of a graph that is dual to the original one. Thus, we can consider dual (to each other) structures of graphs, or in other words, mutually dual classes of isomorphic graphs (they consist of pairwise mutually dual individual graphs). In particular, mutually dual classes of isomorphic graphs consist of equal amounts of individual graphs.

Let us turn back to the graphs on four vertices. We need to count the graphs which have 4, 5, or 6 edges. It appears that each of these graphs is dual to one of the graphs with 2, 1, or 0 edges respectively, and we have already considered the latter. As we have found, they compose 4 classes of isomorphic graphs (they correspond to the structures (1), (2), (3), and (4) in Fig. 7.6). Therefore, the remaining graphs also comprise 4 classes of isomorphic graphs. In Fig. 7.6, their structures are enumerated with (8), (9), (10), (11). The following pairs of structures are dual to each other: (1) and (8), (2) and (9), (3) and (10), (4) and (11).

Now, we are enabled to answer several essential questions concerning the graphs on four vertices. Actually, we know everything about these graphs now. They comprise 11 classes of isomorphic graphs. The structures of these graphs are shown in Fig. 7.6. We know the amount of graphs in each class as well. We summarize these results in the following table reftab424.

Table 7.1. Amount of graphs in the corresponding class

No. of structure	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Amount of graphs	1	6	3	12	4	4	12	1	6	3	12

Summing up the numbers in the second row, we get the amount of all different graphs that can be constructed on 4 vertices. It appears that there are 64 such graphs.

The table reveals that the classes of isomorphic graphs differ greatly by the number of graphs composing them. Therefore, the relation between the number of all graphs (64) and the number of classes of isomorphic graphs is very complex. Needless to say, this complexity exacerbates with the increase of the number of vertices.

We emphasize that all the above calculations concerning the graphs on four vertices have been carried out only to illustrate the complexity of the problem of finding the amount of classes of isomorphic graphs. As to the determining the amount of all graphs constructed on four given vertices, such a problem can be solved with the application of elementary combinatorial tools, which we have familiarized ourselves with extensively thanking the number of problems solved in previous chapters.

So, let us be required to answer the question: how many different graphs can be constructed on four vertices a, b, c, and d? Each such graph can either have the edge joining the vertices a and b or not. The same concerns to the edges ac, ad, bc, bd, cd. It appears that we

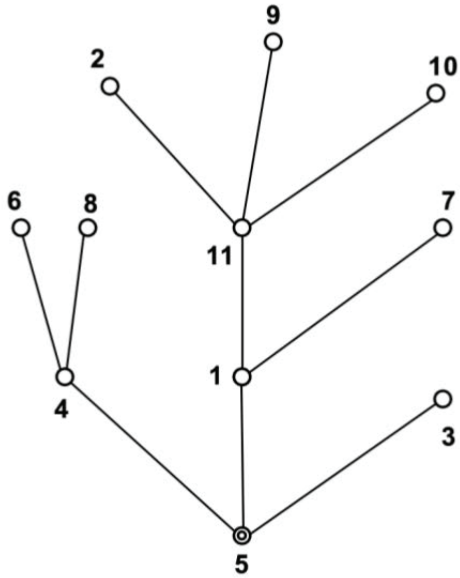

Figure 7.7. Rooted tree on eleven vertices.

get on or another graph if for every of six pairs of vertices – $< a;b >$, $< a;c >$, $< a;d >$, $< b;c >$, $< b;d >$ and $< c;d >$, we decide whether to join the corresponding vertices with an edge or not. In addition, our decision (to join or not to join) concerning any pair does not depend on the decisions made in respect of the other five pairs. Clearly, we are in a "classical" situation where the combinatorial rule of product is applicable. Therefore, we conclude that there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$ different graphs on 4 vertices.

5. Now, we are getting down to the question about the number of all rooted trees that can be constructed on n vertices. This problem is much more complex than the similar problem concerning all graphs on the same number of vertices. We will proceed as follows. First, we will figure out a way to encode all rooted trees in such a manner that there will be a bijection between the objects of both types (rooted trees and their codes). Then we will ensure that the construction of codes is simple enough for them to be easily countable. In other words, we intend to exploit the principle of equality of amounts of objects between which there exists a bijective correspondence. To this end, we will find objects (codes) which, on the one hand, are in bijective correspondence with rooted trees, and on the other hand, are easily countable. As we remember, such an approach has proved useful on many occasions. For instance, recall the way we dealt with the shortest paths connecting opposite vertices of a rectangle.

Let us denote n vertices on which our rooted trees will be constructed by the initial n natural numbers: $1, 2, 3, 4, \ldots, n-1, n$. We will ensure shortly that such denotation is very convenient for the realization of our plan.

Now, we have to describe the procedure of the creation of the code of a rooted tree. We begin with two examples explaining it. Both examples are supplemented with the corresponding generalization.

Example 7.1. *In Fig. 7.7, there is the rooted tree on eleven vertices denoted by numbers according to the above suggestion:* $1,2,3,4,5,6,7,8,9,10,11$. *Its root is the vertex 5, and leaves are the vertices* $6,8,2,9,10,7$ *and* 3. *Below, we outline the algorithm of construction of the code.*

First Stage. Find the leaf vertex with the least number. In our case, this is the vertex 2. Descend from it to the root of the tree (we have ensured above that such descend can be performed uniquely), taking note of all vertices that we pass on our way: $2,11,1,5$. Replace the obtained descending branch with the reverse ascending one $(5,1,11,2)$ and remove the last vertex (the leaf) from it. We get the first block of the future numeric code of the tree: $5,1,11$.

After that, we determine the leaf that is denoted by the least number among the remaining ones. In our case, this is the vertex 3. Construct the maximal descending branch unrelated with the one constructed in the previous step: $3,5$. Again, we rewrite it in the reverse order $(5,3)$ and remove the last vertex (the leaf). The resulting chain is the second block of the code of our tree. In the present case, this is the chain of length 1 consisting of the vertex 5 only.

The next leaf is the leaf 6. The maximal descending branch for it which is unrelated to the two previously constructed branches is $6,4,5$. Therefore, the corresponding block of the future code is $5,4$.

We proceed in a similar fashion with the leaf 7. Contrary to the previous branches, its bottom vertex is not the root of the tree but the vertex 1 laying on the branch constructed above (in the first step) for the leaf 2. Thus, the descending branch (unrelated to the previous ones) for the vertex 7 has only two vertices: 7 and 1. Hence, it delegates block 1 to the code of the tree.

Repeating the above procedure with the remaining leaves $8,9$ and 10, we find the blocks corresponding to them. These are the blocks $4,11$ and 11.

Second Stage. Construct the table 7.2. In the first row, there are all leaves of the tree in increasing order, and in the second row, there are their respective contributions to the code of the tree (that is, the blocks of numbers corresponding to the leaves according to the rule stated above).

Table 7.2. Blocks of numbers corresponding to leaves

Leaf vertex	2	3	6	7	8	9	10
Block of the code	5, 1, 11	5	5, 4	1	4	11	11

Removing the "barriers" separating the blocks in the bottom row of the table, we get the code of the rooted tree in question:

$$5,1,11,5,5,4,1,4,11,11. \tag{7.1}$$

Basing on this special case of the tree on eleven vertices, we can outline several essential properties of its code.

First, the number of blocks forming the code coincides with the number of the leaf vertices of the rooted tree. Also, it coincides with the number of mutually unrelated branches which cover the entire tree (the union of branches taken assets of vertices contains all vertices of the tree). These facts directly result from the algorithm of the construction of the code and from the property of mutually unrelated branches of the tree which contain all its vertices that we have proved above. We defer the consideration of the problem of splitting the code into blocks until the moment when we will discuss the "decryption" of the code and restoring the respective rooted tree.

Secondly, the length of the code is less than the number of vertices of a tree by 1. In particular, the length of the code of the tree on eleven vertices is 10. In other words, the length of the code of a rooted tree equals the number of its edges. Why? Because there are as many instances of any vertex of a tree in its code as there are ascending edges stemming from this vertex. This is the result of the fact that the blocks of a code supplemented by the corresponding leaves compose the set of pairwise mutually unrelated branches, which covers the entire tree.

Thirdly, the first number of the code of a tree indicates its root.

Basing on the structure of the code and its abovementioned properties, we have to find out if a code uniquely defines the corresponding tree. First of all, we need to determine how to split a code of $n-1$ initial natural numbers into several constituents, which we call blocks. The first block is a leaf-to-root branch without a leaf. Therefore, all numbers in it are different. However, the first number of the second block is inevitably one of the numbers comprising the first block, because descending from the second leaf to the bottom, we stopped at the moment we reached the vertex of the leaf-to-bottom branch that we had already followed. These considerations apply to all other blocks as well: the third, fourth, etc. Every block begins with the number already available in at least one of the previous blocks and ends right before the number which initiates the next block (and as such this number is also already available in our code). Therefore, any code can be uniquely split into separate blocks.

Let us take a look at the code (7.1) which we derived applying the above algorithm to the rooted tree in Fig. 7.7. We ignore how this code was constructed and attempt to split it into blocks applying the considerations of the previous paragraph.

The code begins with the number 5 (the root of the tree), followed by the new numbers 1 and 11. The next component of the code is 5 again. This signals that the previous number 11 concludes the first block. Here it is: $5, 1, 11$. The number 5 which is in the fourth position in the code begins the second block. And effectively completes it, as it is followed by another instance of the number 5, which arises for the third time in the code. Thus, the second block consists of the number 5 alone. The next 5 (sitting in the fifth position) opens the new, the third, block. It is continued with the number 4 in the sixth position. And this is the last number in the third block because the next number is 1, which we have come across earlier (in the first block). Therefore, we have the two-element third block: $5, 4$. All remaining numbers of the code (positions from the seventh to tenth) are not new, hence, all other blocks (fourth to seventh) are composed of one element each. Here are these blocks: $1; 4; 11; 11$. We have ensured that our rule of decryption of the code into separate blocks is correct. Applying it to the code (7.1), we get the very same blocks that were used to create it. In other words, basing on the algorithm of the construction of the code, we

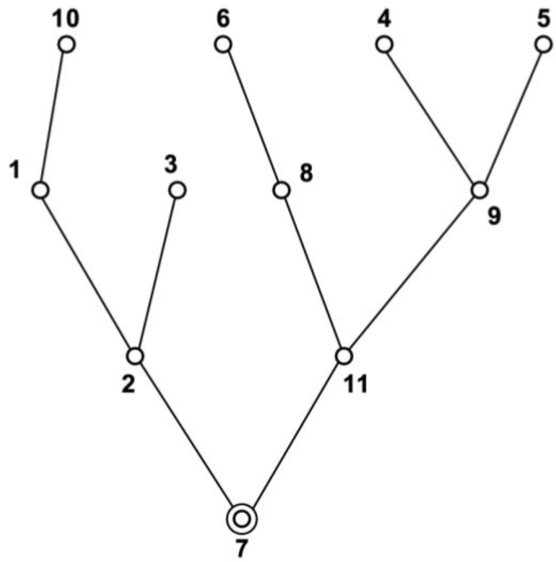

Figure 7.8. Another rooted tree on eleven vertices.

were able to construct the reverse algorithm of decomposition of the code into blocks. In particular, we have discovered a very important fact: not only do the blocks uniquely define the code (which is quite obvious as we just attach blocks to each other) but also the code uniquely defines the blocks from which it has been constructed. In our example, we have the following decomposition:

$$5,\ 1,\ 11|5|5,\ 4|1|4|11|11. \tag{7.2}$$

What our actions should be to restore a rooted tree by the decomposition of its code? Having determined the length of the code (the amount of numbers composing it), add 1 to the number obtained. The result is the number expressing the amount of vertices in the corresponding tree. In our case, this is the number 11, as the code is of length 10. As we remember, in order to make the proposed encoding system for the rooted trees convenient, we decided to denote the vertices of trees with initial natural numbers. Thus, in our case (for the code (7.1)), the vertices will be the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Write down in ascending order those of these eleven numbers which are absent in the coding sequence (7.1). These are the following numbers: $2, 3, 6, 7, 8, 9, 10$. These are the leaf vertices of the tree. Placing them in the above order at the end of each block of the decomposition of the code (7.2), we get the complete collection of pairwise mutually unrelated branches of the rooted tree. This collection contains full information about the edges of the tree. As the root of the tree is also known (the first number of the code), we arrive at the conclusion that the code uniquely defines the rooted tree.

Example 7.2. *In Fig. 7.8, there is another tree on 11 vertices with the root* 7. *What is required to do to create its code? There are* 5 *leaves in the tree, hence, its code is composed*

of 5 blocks, each being an ascending branch of the tree with leaf removed. Let us construct them one by one, placing the corresponding leaf vertices in ascending order.

In this case, the leaves should be arranged as follows:
$3,4,5,6,10.$
First, construct the maximal branch with the leaf 3. Arrange its vertices in line with the growth of their levels:
$7-2-3.$
Removing the last vertex (the leaf 3), we get the first block of the code:
$7,2.$
Now, let us move to the next leaf, the leaf 4. We descend from it along the edges until we reach the vertex which is included in the previous block. In our case, this is the vertex 7 (the root). Write down the vertices of this branch bottom (vertex 7) to top (vertex 4):
$7-11-9-4.$
Detaching the last vertex, we get the second block of the code:
$7,11,9.$
It comes to the turn of the leaf 5. Descending from it, we immediately arrive at vertex 9, which is included in the latter block. This is the stop sign. We have the branch
$9-5$
and the block
9
of only one number. Similarly, for the vertex 6, we construct the branch
$11-8-6$
and the block
$11,8,$
and for the vertex 10, we get the branch
$2-1-10$
and the block
$2,1.$
Then, line up the numbers of all blocks, adhering to the order in which the blocks have been constructed, to get the code of the given rooted tree:
$7,2,7,11,9,9,11,8,5,1.$
In order to decrypt the code and construct the rooted tree, we need to repeat the encoding procedure in reverse order. While considering Example 1, we have performed both the procedure of encoding and the reverse problem of decryption. We are not going to run the reverse procedure concerning the latter code of the tree shown in Fig. 7.8. Instead, we will decrypt several codes of the rooted trees concerning which we have no preliminary knowledge (provided by the graphical illustration in the previous examples). This would be much more interesting and instructive.

Example 7.3. *Construct a rooted tree, decrypting its code*
$10,7,5,11,7,5,7,10,10,5,10.$

To begin with, pay attention to the fact that the problem is well-posed. Indeed, we agreed to denote the vertices of trees with consecutive natural numbers. In addition, we know that the length of a code should be less than the number of vertices by 1. This means

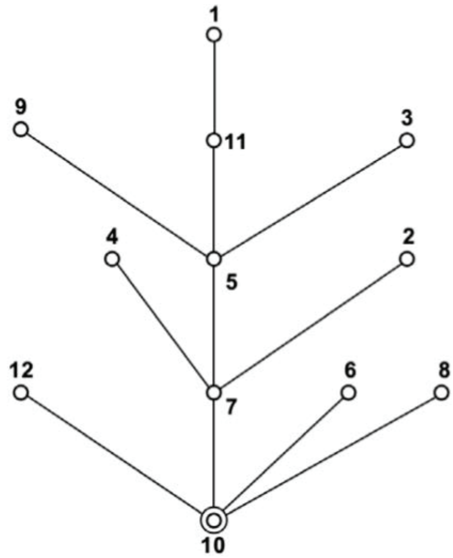

Figure 7.9. Geometric scheme of the rooted tree.

that a tree can not have vertices denoted with numbers greater than $n+1$, given that the length of its code is n. We are provided with the code of length 11. It should contain no numbers greater than 12. And this holds. Therefore, the code is "good". Let us get down to decryption.

The root of the tree is the vertex 10, which begins the code. The numbers $10, 7, 5$, and 11 compose the first block of the code. We should stop at the number 11, because the next number is 7, which is the second instance of this number, and so it opens the second block. This block ends with the very same number 7, as it is followed by the number 5 which we have come across before and which begins the next (third) block. Similar to the second one, it consists of only one number, because the next number is again the repetition. And this property is inherent to all further blocks. All blocks, except the first, are composed of one number. Hence, we derived the following decomposition of the code into blocks:

$$10,\ 7,\ 5,\ 11|7|5|7|10|10|5|10$$

$1, 2, 3, 4, 6, 8, 9, 12$

(recall: the number of vertices of a tree is greater than the amount of numbers in its code by one). Extend the first block with the leaf 1, the second with the leaf 2, the third with the leaf 3, and so on up to the last block which is extended with the leaf 12. We have the collection of branches of the wanted tree. Here are these branches (the lines denote the corresponding edges):

$10-7-5-11-1; 5-3;$

$7-4; 10-6; 10-8; 5-9; 10-12.$

Combining branches appropriately, we get the geometric scheme of the rooted tree. It is depicted in Fig. 7.9.

Example 7.4. *Decrypt the code*

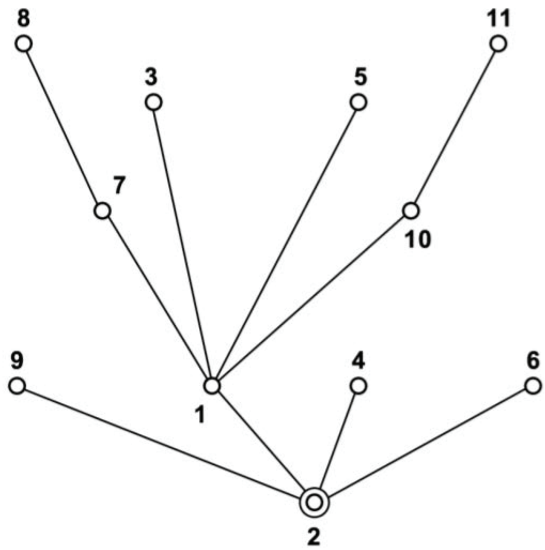

Figure 7.10. The corresponding rooted tree.

$2,1,2,1,2,1,7,2,1,10$
and draw the scheme of the corresponding rooted tree.

The sought tree should have 11 vertices, namely:
$1,2,3,4,5,6,7,8,9,10,11.$

This is because the code contains 10 numbers (positions). As all numbers of the code are less than 12, the problem is well defined.

The root of the tree is the vertex 2. Applying the familiar algorithm (each next block starts with the number that has already appeared in the code before, and conversely, every such number opens the new block), we split the code into blocks:

$$2,\ 1|2|1|2|1,\ 7|2|1,\ 10$$

(neighboring blocks are separated with vertical lines).

Next, arrange the leaf vertices of the wanted tree in ascending order (those vertices that do not participate in the construction of the code):
$3,4,5,6,8,9,11.$

Attaching them to the corresponding blocks (blocks in the same positions), we get the complete set of pairwise unrelated branches:
$2-1-3, 2-4, 1-5, 2-6, 1-7-8,$
$2-9$ and $1-10-11.$

The scheme of the wanted tree is shown in Fig. 7.10.

Example 7.5. *Build the scheme of a rooted tree by its code:*

$$10,8,7,1,5,4,4,5,1,7.$$

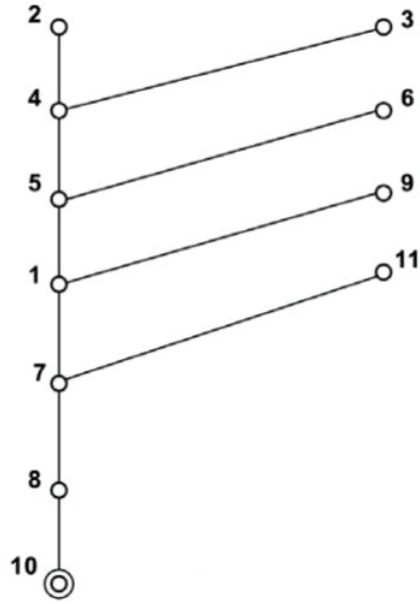

Figure 7.11. The scheme of the rooted tree.

Decompose the code into blocks:

$$10,\ 8,\ 7,\ 1,\ 5,\ 4|4|5|1|7.$$

Arrange the numbers which are absent in the code in ascending order (the greatest of them is 11, *as the length of the code is* 10*):* $2, 3, 6, 9, 11$.

The branches of the tree are

$10-8-7-1-5-4-2; 4-3; 5-6;\ 1-9$ *and* $7-11$.

Combine them to get the scheme of the rooted tree (see Fig. 7.11).

Let us summarize our reasonings. We developed an algorithm to encode rooted trees with certain numeric sequences. If a tree has n vertices denoted by natural numbers from 1 to n, then its code contains some of these numbers (the numbers in a code may repeat), and their overall amount (the length of a code) is $n-1$. In addition, any tree has a unique code corresponding exclusively to it. Conversely, we can recover a tree by its code.

The above has the following meaning. Let $D_{kop.}(n)$ be the set of all rooted trees that can be constructed on the vertices $1,\ 2,\ 3,\ \ldots,\ n$, and $K_{n-1}(n)$ be the set of all collections of $n-1$ numbers (among which there can be the same numbers) from the set $\{1, 2, 3, \ldots, n\}$. Our rule of encoding of rooted trees establishes a bijection between the sets $D_{kop.}(n)$ and $K_{n-1}(n)$. In particular, both sets contain equal amounts of elements. The elements of the second set are easily countable. Really, this set is composed of $(n-1)$-element sequences (or collections), every component of which can gain natural values from the interval $[1, n]$. The fact that the value of any component does not depend on the values of other components completes the picture. Therefore, by virtue of the combinatorial rule of product, the set

$K_{n-1}(n)$ (the set of all codes of n-vertex trees) contains n^{n-1} elements. The same amount of elements is in the set $D_{kop.}(n)$. Hence, on n give vertices, one can construct

$$n^{n-1}$$

different rooted trees.

We have remarked earlier that on given n vertices, there can be constructed n times more rooted trees than unrooted. Thus, there can be constructed

$$n^{n-2}$$

unrooted trees on n vertices.

7. There are many other methods to count the number of trees on n vertices. In this section, we consider one more approach based on a similar idea. This time, we will exploit a bijection between regular (unrooted) trees on n vertices and their codes, which are the sequences of natural numbers of length $n-2$. This algorithm was suggested by German mathematician G. Profer around the 1900s.

Again, assume the vertices of a tree (this time, this is a regular or unrooted tree) are denoted with consecutive natural numbers $1, 2, 3, ..., n$. We construct the code of a tree in the following way. First, we search among pendant vertices (we have already learned that such vertices are always available) the one with the lowest number. There is only one edge incident to this vertex. The other endpoint of this edge is some vertex of the tree which shall be denoted by k_1. We declare this number the initial number of the future code. Then, we remove the used pendant vertex from the tree (the one that has the lowest number among all pendant vertices of the tree) and the edge incident to it. The residual graph is definitely a tree (as it is connected and there are no cycles in it because there is no reason for them to appear). This tree has one vertex and one edge less than the initial one.

The new tree is to be dealt with similarly. First, find the pendant vertex with the lowest number of all pendant vertices. Then, determine the vertex joined with the one found above. Let it be the vertex k_2. The number k_2 is declared to be the second component of the future code, and the used pendant vertex along with the edge incident to it are removed from the tree. The above procedure is repeated for the new tree which has $n-2$ vertices and $n-3$ edges.

As we can see, there are grounds to call the algorithm of the construction of a code cyclical. The first cycle is composed of the following consecutive actions: searching for the pendant vertex with the lowest number, determining the vertex k_i adjacent to it, removal of the pendant vertex with the lowest number, and the edge joining it with the vertex k_i. In $n-2$ steps, we get the following sequence of $n-2$ numbers

$$k_1, k_2, k_3, ..., k_{n-2}. \tag{7.3}$$

We call his sequence the code of our n-vertex tree. After the last, $(n-2)$-th cycle, we get a two-vertex (a dumbbell) tree. One of its two vertices is k_{n-2}.

At least two remarks are necessary for respect of the above procedure.

First, the numbers of a code are not necessarily different. Some of them can be equal. Moreover, all numbers of a code can be the same. Thorough inspection of the algorithm evidence that any vertex x (expressed with some number) repeats $\rho(x)-1$ times in a code,

where $\rho(x)$ denotes the degree of the vertex x. Indeed, building a code, we remove the edges of a tree one by one in a certain order, until there remains only one edge which joins two pendant vertices. While doing so, according to the algorithm, we remove the edge that joins a pendant vertex with a non-pendant one. The former vertex is simply removed, while the latter becomes the component of a code, preserving its status as a vertex of the tree but having its degree decreased by 1. During the construction of a code, every vertex inevitably becomes pendant, thus its degree turns to 1, and the vertex contributes to the code exactly in the way discussed above in this paragraph.

Second, we called the chain of numbers (7.3) the code of a tree, without giving a proper justification to this term. Such name is appropriate only if a bijective correspondence can be established between all trees on n vertices $1, 2, 3, ..., n$ and all sequences (7.3) of length $n-2$, the components of which are the numbers $1, 2, 3, ..., n$. Currently, we have an injection of the set of trees on the set of sequences (7.3). Therefore, we have to ensure that this correspondence is a bijection. To this end, it suffices to prove that it is a surjection, that is, each sequence (7.3) corresponds to a certain tree according to the stated rule.

So, let us have an arbitrary sequence

$$k_1, k_2, k_3, ..., k_{n-3}, k_{n-2}, \tag{7.4}$$

composed of some of the numbers

$$1, 2, 3, 4, ..., n-1, n. \tag{7.5}$$

We will attempt constructing the tree on vertices (7.5) the code of which is sequence (7.4). Clearly, the algorithm of construction of a tree (and it should dictate which vertices are to be joined with edges) has to be reversed to the algorithm of construction of the code of a tree, which we have discussed above. Let us begin.

First Step. Among the numbers (7.5), we find the lowest number which is absent in sequence (7.4). Remark that such search will always be successful, as sequence (7.4) has fewer members than set (7.5) has numbers. Let s_1 be this number. We join the vertices s_1 and k_1 with an edge, and then remove the first element k_1 of sequence (7.4) and the number $s_1$from set (7.5). This concludes the first step of the construction of a tree. The first edge $(s_1; k_1)$ has appeared on the graph, and instead of sequence (7.4) and set (7.5) we now have the sequence

$$k_2, k_3, k_4, ..., k_{n-3}, k_{n-2} \tag{7.6}$$

and the set

$$1, 2, ..., s_1-1, s_1+1, ..., n-1, n \tag{7.7}$$

respectively.

Second Step. We are going to deal with sequence (7.6) and set (7.7) in the same manner as we have dealt with sequence (7.4) and set (7.5). Let us find the lowest number of set (7.7) which is absent in sequence (7.6). Call it s_2. Join the vertices s_2 and k_2 with an edge, and shorten sequence (7.6) and set (7.7) by removing k_2 and s_2 respectively.

Repeating the above procedure step by step, draw the edges

$$(s_1; k_1),\ (s_2; k_2),\ (s_3; k_3),\ \dots,\ (s_{n-3}; k_{n-3}),\ (s_{n-2}; k_{n-2}).$$

After that, sequence (7.4) is depleted, and there are only two numbers remaining in set (7.5): k_{n-2} and another one which we denote by s. Again, join these two vertices with the edge $(s; k_{n-2})$. This completes the construction of the tree. Although, it remains to ensure that the constructed graph is actually a tree.

To this end, take a careful look at the numbers

$$s_1, s_2, s_3, \dots, s_{n-3}, s_{n-2}, k_{n-2}, s. \tag{7.8}$$

All of these numbers are different. In other words, this is a certain permutation of the numbers of (7.5). Being the vertices of a graph, some of them are joined with edges. Let us research the placement of these edges, moving from the end of sequence (7.8) to its start. First, there are edges $(k_{n-2}; s)$ and $(s_{n-2}; k_{n-2})$, hence, we have the tree

$$s_{n-2} - k_{n-2} - s \quad (T_{n-2})$$

with the vertices s_{n-2}, k_{n-2} and s. We reach the number (vertex) s_{n-3} of sequence (7.8). It is joined with the number k_{n-3} with the edge. According to our algorithm of decryption of a code, the latter number is not one of the numbers $s_1, s_2, s_3, \dots, s_{n-3}$. Thus, this is one of three numbers which are located after s_{n-3} in sequence (7.8): s_{n-2}, or k_{n-2}, or s. Whatever this number is, if we attach the vertex s_{n-3} and the edge $(s_{n-3}; k_{n-3})$ to the tree (T_{n-2}), then we again get the tree constructed on four last vertices of sequence (7.8). Denote this tree by T_{n-3}.

Needless to say, our further investigation will develop cyclically by attachment of new numbers s_i in each step. In order to ascertain in this, let us describe one more step. Consider edge $(s_{n-4}; k_{n-4})$. The number k_{n-4} can not be one of the numbers $s_1, s_2, s_3, \dots, s_{n-4}$ (by their construction), hence, it should be equal to one of the numbers $s_{n-3}, s_{n-2}, k_{n-2}$ or s. Thus, k_{n-4} is one of the vertices of the tree T_{n-3}. Attaching the new vertex to this tree by an edge, we get another tree again, which we denote by the symbol T_{n-4}. Moving further cyclically, we will reach the tree T_1 which contains all the vertices $1, 2, 3, \dots, n$.

It appears that every tree on the vertices $1, 2, 3, \dots, n$ have an $(n-2)$-element numeric code composed of some of the numbers $1, 2, 3, \dots, n$ corresponding to it. Conversely, any code of the above type (sequences of length $n-2$ consisting of the numbers $1, 2, 3, \dots, n$) defines a certain tree with the vertices $1, 2, 3, \dots, n$. These two injections evidence that, in fact, there is a bijection between two sets: the set of different trees on the vertices $1, 2, 3, \dots, n$ and the set of all $(n-2)$-element sequences composed of these numbers. In particular, the amounts of objects of both types (trees and their codes) are the same. The number of codes is easily quantifiable (there are n options for each position, and numbers in different positions can be combined randomly). By the combinatorial rule of product, there are n^{n-2} of them. Hence, this is the amount of different n-vertex trees.

Problems

Problem 7.1. *The sum of degrees of all vertices of any graph is an even number. Why?*

Problem 7.2. *A total graph has n vertices. How many edges does it have?*

Problem 7.3. *Two graphs on the same vertices are considered different if one of them has at least one edge that the other graph does not. How many different graphs can be built on n given vertices?*

Answer. $2^{C_n^2}$. *Any two vertices can be either joined with an edge or not.*

Problem 7.4. *How many different graphs with n vertices and k edges are there?*

Answer. $C^k_{C_n^2}$.

Problem 7.5. *A graph is called directed if the edges AB and BA are considered different. In this case, any two vertices can be joined with one of two different edges (AB or BA) or with both, or not joined at all.*

1. *How many different directed graphs can be constructed on n given vertices?*
2. *In how many of them any two vertices are joined with at least one edge?*
3. *How many n-vertex directed graphs have any two of their vertices joined by one (directed) edge?*
4. *How many directed graphs do not have double edges (any two vertices are either not joined or joined with only one edge)?*

Answer. *1)* $4^{C_n^2}$; *2)* $3^{C_n^2}$; *3)* $2^{C_n^2}$; *4)* $3^{C_n^2}$.

Problem 7.6. *A directed graph is called symmetrical if any two vertices are either joined with two edges or not joined by any. How many such graphs can be constructed on n given vertices?*

Problem 7.7. *If an (undirected) n-vertex graph has n or more edges, then there are cycles in it (at least one). Prove it.*

Problem 7.8. *If every vertex of a graph has degree 1, then the graph has an even number of vertices. Why?*

Problem 7.9. *Let every vertex of a* $2n$*-vertex graph has degree 1. How many connected components are there in the graph?*

Problem 7.10. *Prove that any tree has at least two leaf vertices.*

Problem 7.11. *If there are two or more connected components in a graph and all of them are trees, then such a graph is called a forest.*

How many edges are there in a forest with n vertices and s components?

Problem 7.12. *Create the Profer codes of the trees shown in Fig. 7.12.*

Answer. *(1) 4, 2, 2, 4, 2; (2) 1, 1, 7, 3, 3, 4.*

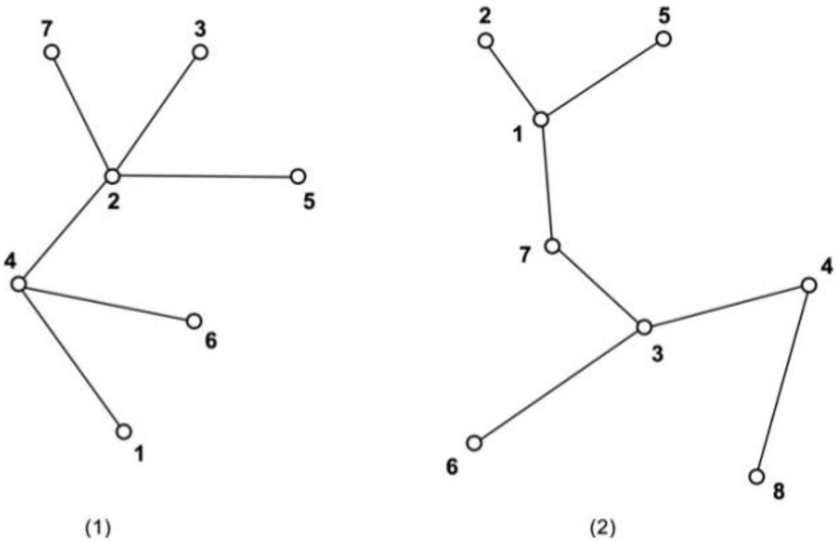

Figure 7.12. Profer codes of the trees.

Problem 7.13. *Construct trees with vertices 1, 2, 3, 4, 5, 6, 7, 8 by their Profer codes. 7, 7, 7, 7, 7, 7; (2) 5, 2, 5, 2, 5, 2; (3) 8, 7, 6, 5, 4, 3; (4) 1, 2, 3, 1, 2, 3.*

Answer. *(1) The tree with edges (1; 7), (2; 7), (3; 7), (4; 7), (5; 7), (6, 7); (8; 7). (2) The tree with edges (1; 5), (4; 5), (7; 5), (3; 2), (6; 2), (8; 2), (5; 2). (3) The tree with edges (1; 8), (8; 3), (3; 4), (4; 5), (5; 6), (6; 7), (7; 2). (4) The tree with edges (4; 1), (7; 1), (1; 2), (5; 2), (2; 3), (6; 3), (8; 3).*

Problem 7.14. *How many different n-vertex trees have only two pendant vertices?*

Answer. $\frac{1}{2} \cdot n!$.

Problem 7.15. *A tree has k vertices with degree 3. Its other vertices are pendants. How many other vertices does the tree have? Prove that any tree of this type has an even amount of vertices. Draw the trees that satisfy the above conditions and have 4, 6, 8, and 10 vertices.*

Answer. $k+2$.

Problem 7.16. *A tree has no vertices of degree 3 or greater. How many pendant vertices are there in such a tree?*

Answer. *2.*

Problem 7.17. *For our convenience, let us call any non-pendant vertex of a tree an inner vertex. Let s be the sum of degrees of all inner vertices of a tree, k is the number of such vertices, and m is the number of pendant vertices. Prove that*

$$m+2k = s+2.$$

Problem 7.18. *A rooted tree is constructed on the vertices 1, 2, 3, 4, 5, 6, 7. 8, 9, 10, 11, 12 and has the following edges: (6; 1), (6; 5), (1; 9), (1; 8), (1; 2), (8; 4), (4; 12), (4; 11), (5; 10), (5; 7), (7; 3). Its root is the vertex 6.*

Draw this tree. Find its leaves and branching points. List all trunk (root-to-leaf) branches. Construct the "branch" code of the above tree.

Answer. Root-to-leaf branches are: 6 – 1 – 9; 6 – 1 – 2; 6 – 1 – 8 – 4 – 12; 6 – 1- 8 – 4 – 11; 6 – 5 – 10; 6 – 5 – 7 – 3. The code of the tree is 6, 1, 6, 5, 7, 1, 5, 1, 8, 4, 4.

Problem 7.19. *Let a rooted tree have the vertices and edges as in the previous problem but its root is the vertex 10 now. Construct its "branch" code*

Answer. 10, 5, 6, 1, 5, 7, 1, 1, 8, 4, 4.

Problem 7.20. *Recover a tree by its code, that is, find its root and edges and sketch it (denote vertices with points, edges with line segments, highlight the vertex which is the root).*

a) 3, 3, 3, 5, 5, 5, 1, 1, 1.4 b) 9, 10, 1, 2, 9, 10, 1, 2, 5; c) 10, 9, 8, 7, 6, 5, 4, 3, 2; d) 6, 1, 6, 1, 6, 1, 6.

Answer. a) The root is the vertex 3. The full list of edges: (3; 2), (3; 4), (3; 5), (5; 6), (5; 7), (5; 1), (1; 8), (1; 9), (1, 10). b) The root is the vertex 9. The full list of edges: (9; 10), (10; 1), (1; 2), (2; 3), (9; 4), (10; 6), (1; 7), (2; 5), (5; 8). c) The root is the vertex 10. The full list of edges: (10; 9), (9; 8), (8; 7), (7; 6), (6; 5), (5; 4), (4; 3), (3; 2), (2; 1). d) The root is the vertex 6. The full list of edges: (6; 1), (6; 3), (6; 5), (6; 8), (6; 10), (1; 2), (1; 4), (1; 7), (1; 9).

Problem 7.21. *On the set of vertices* $\{1, 2, 3, ..., 2n\}$*, various graphs are constructed with all their vertices having the degree of 1. How many different graphs of this type exist?*

Answer. $\frac{(2n)!}{2^n \cdot n!}$*, or equivalently,* $(2n-1)!!$ *(the product of all consecutive odd numbers from 1 to* $2n-1$*).*

Solution. First Approach. A graph is a forest composed of two-vertex trees. Therefore, the problem can be restated as follows: how many ways are there to split a $2n$-element set into n two-element subsets? The decomposition can be constructed in the following manner. First, choose a pair for the number 1. There are $2n-1$ options, as any of the other $2n-1$ numbers suit. If the choice is made, and the pair for the number 1 is the number k_1, then we take the lowest of the remaining numbers, denote it by s_2, and choose a pair for it. There are $2n-3$ candidates for this role. Thus, whichever the first pair $(1; k_1)$ is, there are $2n-3$ possibilities for the second pair $(s_2; k_2)$. Assume we have chosen one of them (the exact choice does not matter). Then, consider the lowest of the numbers which have not been paired yet and denote it by s_3. There are $2n-5$ options to choose the component k_3 to create a pair with s_3, and so on. Clearly, our procedure of pairing produces different partitions of the initial set into two-element subsets whenever at least one number k_i is chosen differently. Therefore, there are $(2n-1)!!$ partitions in total.

Second Approach. This time, we construct the partition as follows. We choose the first two-element subset. There are C_{2n}^2 different ways to make this choice. There are C_{2n-2}^2 options for the second two-element subset, C_{2n-4}^2 for the third, and so on, up to the last one for which there are C_2^2 ways to choose it. According to the combinatorial rule of product, the entire procedure can be performed in

$$C_{2n}^2 \cdot C_{2n-2}^2 \cdot \ldots \cdot C_2^2 \tag{7.9}$$

ways. The partitioning processes differ either by the availability of non-identical subsets or by the order in the identical subsets appear. In the latter case, the resulting partition is the same: the set is split into the same subsets and the only difference is in the order in which these subsets have been created. As n subsets can be ordered in $n!$ ways, any partition is accounted for $n!$ times in expression (7.9). Hence, the correct answer to the problem is the number

$$\frac{1}{n!} \cdot C_{2n}^{2} \cdot C_{2n-2}^{2} \cdot C_{2n-4}^{2} \ldots C_{2}^{2},$$

which is equivalent to

$$\frac{(2n)!}{2^n \cdot n!}.$$

Third Approach. First, choose n of the available $2n$ numbers and arrange them in some order (the ordering principle is irrelevant), say, increasingly:

$$i_1, i_2, i_3, \ldots, i_n.$$

There are C_{2n}^{n} ways to make it. The remaining numbers are to be put in a certain order as the second row under the numbers i_k $(k = 1, 2, \ldots, n)$. We get the table with two rows:

$$\begin{array}{ccccc} i_1 & i_2 & i_3 & \ldots & i_n \\ j_1 & j_1 & j_3 & \ldots & j_n. \end{array}$$

It contains all $2n$ original numbers. Whichever the first row is, there are $n!$ ways to create the second row. Therefore, there are $C_{2n}^{n} \cdot n!$ such tables in total. If we join two corresponding numbers from different rows of a table with edges (i_1 with j_1, i_2 with j_2, etc.), then we get a graph of the type described in the statement of the problem. Obviously, no graph will be skipped by such a procedure. However, the correspondence between the tables and the considered graphs is not a bijection. Switching the numbers of the same column of a table, we get different tables, but the graph that corresponds to these tables is the same. The numbers of the top row can be replaced with the ones of the bottom row (which is interpreted as a half-turn of dumbbells) in any subset of the columns of a table. There are 2^n such subsets (the number of subsets of an n-element set). Therefore, any wanted graph is represented by 2^n tables. In other words, there are 2^n times fewer graphs of interest than there are tables. This means that there are

$$\frac{C_{2n}^{n} \cdot n!}{2^n}$$

graphs. Obviously, this is the same number as before.

Problem 7.22. *Given vertices* $1, 2, 3, \ldots, 3n-1$*, we are required to construct a forest of n isomorphic trees of the scheme* $x - y - z$*. How many ways are there to perform this?*

Answer. $\frac{(3n)!}{2^n \cdot n!}$.

Solution. A forest of the stated type might be constructed as follows. Create some permutation of the numbers $1, 2, 3, 4, \ldots, 3n-1, 3n$. Moving from one end of the permutation to the other, we split it into blocks of three numbers each. Every such block abc has a tree $a - b - c$ with edges $(a; b)$ and $(b; c)$ corresponding to it. Hence, the entire permutation

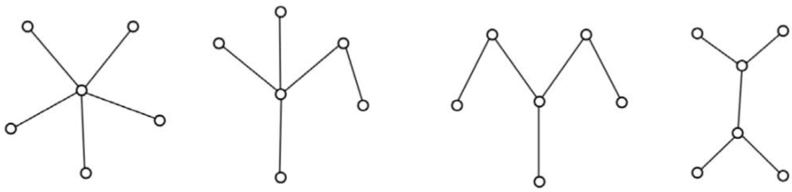

Figure 7.13. Structures of classes of isomorphic graphs on six vertices.

has a forest of similar $x-y-z$ trees corresponding to it. The above correspondence is not bijective. The number of permutations is known. To use it to answer the question of the problem, we need to determine how many different permutations correspond to the same tree by the introduced rule. This is a rather easy exercise. Consider a permutation. Splitting it into blocks, we get a forest, as it has been shown above. Which transformations of the permutation leave the forest unchanged? There are two types of transformations. First, it is obvious that the forest remains unchanged upon the switch of the first and third numbers in several (from 1 to n) blocks. Also, it will not change if we randomly rearrange the blocks. There are 2^n ways to switch the first and third blocks, and $n!$ ways to rearrange the order of blocks. In addition, the rearrangement of blocks can be arbitrarily combined with the mentioned changes inside them. By virtue of these transformations (including the identical one), from the original permutation of $3n$ numbers, we get $2^n \cdot n!$ permutations which have the same forest corresponding to them. Thus, there are

$$\frac{(3n)!}{2^n \cdot n!}$$

different forests of the stated type.

Remarks. 1. The methodology presented above can be applied to the previous problem. We encourage the reader to attempt this.

2. There is a different approach to Problem 7.22. It is rewarding to find and apply it.

Problem 7.23. *On the vertices* $1, 2, 3, \ldots, kn-1, kn$ *(k and n are given natural numbers), a forest is to be built which is composed of n isomorphic trees of the type* $x_1 - x_2 - x_3 - \ldots - x_k$ *(that is trees-chains of length k). How many ways are there to perform this?*

Answer. $\frac{(kn)!}{2^n \cdot n!}$.

Hint. This problem can be solved using the approach suggested in the previous problem.

Problem 7.24. *In Fig. 7.13, the structures of classes of isomorphic graphs on six vertices 1, 2, 3, 4, 5, 6 are shown.*

How many graphs are there in each class?

Answer. a) 6; b) 120; c) 360; d) 90.

Problem 7.25. *How many ways are there to build a forest composed of two trees on 5 vertices?*

Answer. 110.

Problem 7.26. *How many ways are there to build a forest composed of two trees on 6 vertices?*

<u>*Answer.*</u> *1080.*

Problem 7.27. *How many ways are there to build a forest composed of two trees on* $2n+1$ *vertices?*

<u>*Answer.*</u>

$$C^1_{2n+1}\cdot(2n)^{2n-2}+C^2_{2n+1}\cdot 2^0\cdot(2n-1)^{2n-3}+C^3_{2n+1}\cdot 3^1\cdot(2n-2)^{2n-4}+$$
$$+C^4_{2n+1}\cdot 4^2\cdot(2n-3)^{2n-5}+\ldots+C^n_{2n+1}\cdot n^{n-2}\cdot(n+1)^{n-1}.$$

Problem 7.28. *It is given that a connected graph has n vertices and n edges. Prove that it has at least three edges, the removal of any one of which transforms the initial graph into a tree.*

Problem 7.29. 1. *If all n vertices of a connected graph are of degree 2, then:*

2. *it has n edges; upon removal of any one of its edges we get a tree. Prove the above statements.*

Problem 7.30. *Consider connected graphs on n given vertices* $(n \geq 3)$ *all of which are of degree 2. How many such graphs are there?*

<u>*Answer.*</u> $\frac{(n-1)!}{2}$.

Problem 7.31. *There is an even amount of vertices with odd degrees in any graph. Why?*

Problem 7.32. *Which is the lowest possible number of vertices in a graph all vertices of which are of degree m?*

<u>*Answer.*</u> $m+1$.

Problem 7.33. *Consider graphs on seven vertices all of which are of degree 2. 1) How many such graphs are there? 2) How many such non-isomorphic graphs are there?*

<u>*Answer.*</u> *1) 465; 2) 2.*

Problem 7.34. *If all vertices of a graph with edges have even degrees, then there is at least one cycle in a graph. Prove it.*

Problem 7.35. *The distance between two vertices of a tree is the number of edges that one needs to pass to get from one of them to another. The distance between the most remote vertices of a tree is called its height. How many different trees of height 2 can be constructed on n vertices* $(n \geq 3)$*? Are all of them isomorphic?*

<u>*Answer.*</u> *n; yes.*

Problem 7.36. 1. *How many trees of height* $n-2$ *can be constructed on n vertices?*

2. *How many of them are pairwise non-isomorphic?*

<u>*Answer.*</u> *1)* $\frac{n!}{2}\cdot(n-3)$*; 2)* $\left[\frac{n-2}{2}\right]$ *(the integer part of the number* $\frac{n-2}{2}$*).*

Problem 7.37. 1. *How many trees of height 3 can be constructed on n vertices* $(n \geq 4)$*?*

2. *How many of them are non-isomorphic?*

Answer. *1)* $(2^{n-2}-2)\cdot C_n^2$. *There is another counting approach which provides the answer in the form* $\frac{1}{2}[C_n^2\cdot 2\cdot(n-2)+C_n^3\cdot 3\cdot(n-3)+C_n^4\cdot 4\cdot(n-4)+\ldots+C_n^{n-2}\cdot(n-2)\cdot 2]$.
2) $2^{n-3}-1$.

Problem 7.38. *Integer-valued points (points that have both their components integer) of the coordinate plane are the vertices of an infinite graph. Its edges are line segments that join the points one of the coordinates of which is the same and the other differs by one. In other words, the vertices of this graph are the nodes of the network on the plane (splitting the plane into cells) and its edges are the sides of the corresponding cells.*

We depart from the point of origin and move along the edges of the graph from one vertex to another: one edge is one step. At every vertex, we can choose any of four directions.

1. *How many vertices we can find ourselves at in n steps?*

2. *How many vertices can be reached not earlier than in n steps?*

Answer. *1)* $(n+1)^2$; *2)* $4n$.

Problem 7.39. *Assume we travel the graph from the previous problem again. Departing from the point of origin and moving along the edges of the graph from one vertex to another (one edge is one step), we are required to return to the point of origin in n steps. For which values of n such journey is possible and how many ways (different paths) are there to perform it?*

Answer. *The journey satisfying the stated conditions is possible only if n is even. If* $n=2k$, *then there are*

$$\sum_{s=0}^{k}\frac{(2k)!}{(s!)^2\cdot((k-s)!)^2}$$

different routes.

Problem 7.40. *If a graph has n vertices and more than* C_{n-1}^2 *edges, then it is connected. Prove it.*

Problem 7.41. *If a (finite, that is the one that has a finite number of vertices) graph has exactly two vertices of odd degrees, then these vertices belong to the same connected component of the graph (in other words, there is a chain joining them). Prove it.*

Problem 7.42. *Let* Γ *be a connected graph, A and B are two of its vertices. The length (the number of edges) of the shortest chain joining A and B is called the distance between the vertices A and B in the graph* Γ. *The distance between the most remote vertices of the graph* Γ *is called its diameter. Denote the diameter of the graph* Γ *by the symbol* $d(\Gamma)$.

1. *The vertices of a graph are the vertices of a cube. Determine the diameter of the graph.*

2. *The vertices of a graph are the vertices of an n-gon, and its edges are the sides of this n-gon. Determine the diameter of the graph.*

3. *The vertices of a graph are the vertices of a pyramid with an $(n-1)$-sided base, and its edges are the edges of this pyramid. Determine the diameter of the graph.*

4. *The vertices of a graph are the vertices of a prism with an $n/2$-sided base (n is even), and its edges are the edges of this prism. Determine the diameter of the graph.*

5. *What does the diameter of a complete (total) graph (any two vertices are connected with an edge) equal to?*

Problem 7.43. *Let Γ be a connected graph, and A is its vertex. Introduce the function $t(A)$, which will denote the distance from the vertex A to a vertex that is the most remote from it. Let V be the vertex (or one of such vertices) of the graph Γ for which the function $t(V)$ reaches its maximum (on the domain of vertices of the graph Γ). The vertex V is called the center of the graph Γ, and the value of the function $t(V) = r(\Gamma)$ is called its radius.*

Determine the centers and the radiuses of the following graphs.

1. *Vertices A_i $(i = 1, 2, ..., 2k+1)$, edges A_iA_{i+1} $(i = 1, 2, ..., 2k)$.*

2. *Vertices B_i $(i = 1, 2, ..., 2k)$, edges B_iB_{i+1} $(i = 1, 2, ..., 2k-1)$.*

3. *The vertices and edges of a graph are the vertices and edges of a pyramid.*

4. *The vertices and edges of a graph are the vertices and edges of a prism with $n/2$-sided base (n is even).*

5. *The vertices and edges of a graph are the vertices and sides of an n-gon.*

6. *The vertices and edges of a graph are the vertices and edges of an octahedron (a bipyramid with a rectangular base).*

Problem 7.44. *Prove that the diameter of a connected graph does not exceed double its radius:*

$$d(\Gamma) \leq 2r(\Gamma).$$

Provide the examples of graphs for which: a) $d(\Gamma) = r(\Gamma)$; b) $d(\Gamma) = 2r(\Gamma)$.

Problem 7.45. *(Cayley's Problem). There are n cities. The construction of the road between any two cities has a certain cost (this cost is different for every pair of cities). A road network is to be created which should connect all cities. The cost of construction should be the lowest possible. How this can be achieved?*

Discussion of the Problem. There are infinitely many possible variations of the present problem. Instead of the cost of road construction, one might consider distances between cities. Road construction context itself might be replaced with the construction of gas or oil pipelines, and so on. The plot of the problem, which is just one of possible interpretations, is of no importance. On the contrary, the mathematical essence of the problem is to be investigated. Let us consider the mathematical interpretation of the problem

There are n points which are the vertices of a future graph. We are provided with a table of prices for drawing every edge. Now, if a graph is constructed, then its price is equal to the sum of prices of its edges (be definition). Our task is to master the art of construction of a graph that possesses all the following properties:

1. its vertices are n given points;

2. it is connected;

3. its price is the lowest possible.

In order to avoid misunderstanding, explicitly assume that the prices of edges are positive.

Analyzing the problem, we come to the conclusion that the sought graph has to be a tree. This results from the fact that a connected graph which is not a tree can be made "cheaper" if an edge is removed which is part of a cycle. As we already know (or, at least, we can easily prove that), the graph remains connected after such procedure, and its price is reduced by the price of edge that is to be removed.

Hence, in our task, the words "construct a graph" might be replaced with the phrase "construct a tree".

The Algorithm of Construction. First Step. In the table that contains all possible edges which can join the given n vertices pairwise along with their prices, we search for the edge with the lowest cost (if there are more than one such edge, then we can choose any of them). Let us chose the edge u_1, the price of which is $w(u_1)$. We emphasize that the prices of all other edges are greater than or equal to the price of the chosen one (for any edge x, the inequality $w(x) \geq w(u_1)$ holds). We draw the edge u_1 (joining the vertices prescribed by the table, of course) and delete it from the list (table).

Second Step is in no way different from the first one. Except for two nuances: firstly, we choose an edge from the table which is shorter by one position. Let it be an edge u_2. Secondly, having drawn the edge u_2, we delete it from the table along with the edge which creates a cycle in combination with the edges u_1 and u_2 (if any).

Third Step. From the new table, we choose one of the edges that have the lowest price (let it be an edge u_3), draw it and delete it from the table along with the edges (if any) which create cycles (separately) in combination with the edges u_1, u_2, u_3 (it is unnecessary that all these edges participate in each cycle).

All further steps are similar to the third one. If the edges $u_1, u_2, u_3, \ldots, u_k$ are already drawn, and following the choice of the last of them we have removed from the table the edge u_k and all edges that create cycles with some of the edges $u_1, u_2, u_3, \ldots, u_k$, then at the $(k+1)$-th step we should act as follows. First, we choose the edge u_{k+1} which has the lowest cost among the edges in the shortened (at the previous step) table, place it between the corresponding vertices of the future tree, and then remove from the table u_{k+1} and those of the remaining edges that create cycles with the previously drawn edges $u_1, u_2, \ldots, u_k, u_{k+1}$.

The last one is the $(n-1)$-th step. The given vertices and the edges $u_1, u_2, \ldots, u_n$ compose the wanted tree which is the one of the lowest overall cost.

Problem 7.46. Additional Exercise. *Provide grounds for the above algorithm. Prove that a tree U constructed with the above algorithm has the lowest cost among all other possible trees.*

Proof. It should be emphasized that the task is not about proving that the tree with the lowest cost is unique. In fact, there can be many such trees. And the above algorithm does not contradict with this fact. As according to it, at each step, we choose the edge with the lowest price and there can be several such edges (or even all of the available).

So, the problem can be restated as follows. Let U be any tree constructed with our algorithm. Its cost is $w(U)$. Assume that V is another tree of the lowest cost. We are required to prove that $w(U) = w(V)$.

During the construction of the tree U, its edges were attached to each in the following order: $u_1, u_2, u_3, ..., u_{n-1}$ (while $w(u_1) \leq w(u_2) \leq w(u_3)... \leq w(u_{n-1})$). Now, we will consider the edges u_i in the very same order, comparing the trees U and V.

Let us begin with the edge u_1. If it is one of the edges of the tree V, then we move on to the edge u_2. Alternatively, if u_1 is not the edge of the tree V, then we act as follows. We embed the edge u_1 into the tree V, getting a graph Γ_1 which has a cycle K_1 which involves the edge u_1. Other edges of this cycle belong to the tree V. All of them have the same cost as u_1. If this was not the case, we would have removed the one of them which had the greater price, and the graph Γ_1 would have turned into a tree with less overall value than V, which would have contradicted with the initial assumption about the tree V.

Removing an edge other than u_1 from the cycle K_1, we get a tree V_1 which shares the edge u_1 with U and has the same price as V: $w(V_1) = w(V)$.

Now, let us find out is there the edge u_2 in V_1. If so, then re-denote this tree by V_2. Otherwise, embed the edge u_2 into the tree V_1. We get a graph Γ_2 with a cycle K_2, containing u_2. In the cycle K_2, there can be no edges with prices greater than $w(u_2)$. If there were, then we could have removed it from K_2, transforming the graph Γ_2 into a tree with the price lower than $w(V)$, which would have contradicted the definition of the tree V. Those edges of the cycle K_2 that do not belong to the tree U can not be of the lower value than $w(u_2)$ as well. If there were an edge t_2 with this property, then building the tree U by our algorithm, we should have chosen it instead of u_2. Thus, all those edges of the cycle K_2 that does not belong to U have the same price as the edge u_2. Removing one of them, we transform the graph Γ_2 into the tree V_2 which has the following properties. Firstly, $w(V_2) = w(V_1) = w(V)$, and secondly, it has at least two shared edges with the tree U: u_1 and u_2. Thorough research of the above considerations concerning the cycle K_2 might reveal a loose end. The entire construction which results in the creation of the tree V_2 is based on the assumption that the cycle K_2 includes at least one edge that does not belong to U. Are there grounds for that to be true? Yes. Indeed, if all edges of the cycle K_2 belonged to U, then the graph U would not be a tree, which is not true.

Thus, we build the trees $V_1, V_2, V_3, ...$ step by step. Any tree V_k of this list has two essential properties. First, it has the lowest value among all trees as $w(V_k) = w(V)$. Second, it has at least k shared edges with the tree U, namely: $u_1, u_2, u_3, ..., u_k$. The list of trees $V_1, V_2, V_3, ...$ can be extended by our rule until there remain edges of the tree U. In other words, the last in the above list is the tree V_{n-1}. For this tree: $w(V_{n-1}) = w(V)$ and$V_{n-1} = U$. Hence, we have proved that $w(U) = w(V)$.

□

Problem 7.47. *Construct trees with the lowest possible overall prices for the following tables of prices of edges (the crossing of the row A_i and the column A_j contains the price of the edge A_iA_j):*

Table 7.3. Table of prices of edges (a)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	x	1	1	1	2	2	2
A_2	x	x	2	1	3	2	3
A_3	x	x	x	1	2	3	1
A_4	x	x	x	x	2	3	3
A_5	x	x	x	x	x	3	2
A_6	x	x	x	x	x	x	4
A_7	x	x	x	x	x	x	x

Table 7.4. Table of prices of edges (b)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
A_1	x	3	2	3	1	2	1
A_2	x	x	3	2	2	3	3
A_3	x	x	x	3	2	1	1
A_4	x	x	x	x	2	1	1
A_5	x	x	x	x	x	1	1
A_6	x	x	x	x	x	x	1
A_7	x	x	x	x	x	x	x

Table 7.5. Table of prices of edges (c)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
A_1	x	1	4	1	4	1	4	1	4
A_2	x	x	2	3	2	3	2	3	2
A_3	x	x	x	1	4	1	4	1	4
A_4	x	x	x	x	2	3	2	3	2
A_5	x	x	x	x	x	1	4	1	4
A_6	x	x	x	x	x	x	2	3	2
A_7	x	x	x	x	x	x	x	1	4
A_8	x	x	x	x	x	x	x	x	2
A_9	x	x	x	x	x	x	x	x	x

Table 7.6. Table of prices of edges (d)

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
A_1	x	1	2	3	4	5	6	7	8
A_2	x	x	7	6	5	4	3	2	1
A_3	x	x	x	1	2	2	4	5	6
A_4	x	x	x	x	5	4	3	2	1
A_5	x	x	x	x	x	1	2	3	4
A_6	x	x	x	x	x	x	3	2	1
A_7	x	x	x	x	x	x	x	1	2
A_8	x	x	x	x	x	x	x	x	1
A_9	x	x	x	x	x	x	x	x	x

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This book is a very good introduction to combinatorics as a first step... This book contains a great deal of problems with answers, examples and exercises. I would recommend it to university instructors who teach combinatorics, to pedagogical universities and colleges students, and also to high school students and first year university students.

Anatoliy Swishchuk
Professor, Applied Mathematics

The book "Combinatorics: First Steps" written by two very experienced mathematicians Mykola Perestyuk and Volodymyr Vyshenskyi is a gem, which will satisfy the needs of all kinds of combinatorics lovers, from high school students to college professors. It has plenty of examples and solved problems that make the digesting of the material easy. It can also serve as a wonderful source of all kinds of combinatorial problems for extracurricular activities in mathematics.

Vyacheslav Futorny, Professor
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